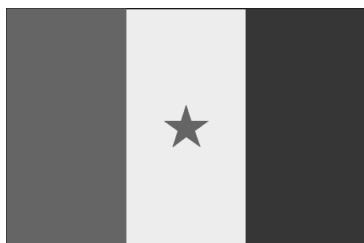


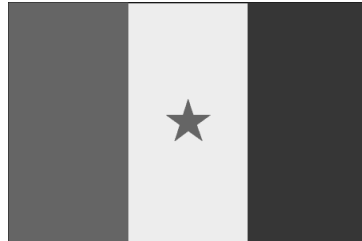
A PUBLICATION FOR THE MIDDLE SCHOOLS OF THE REPUBLIC OF SENEGAL
APPROVED BY THE MINISTRY OF EDUCATION



Introduction to Algebra and Geometry

Grades 9 and 10

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9 and 10**

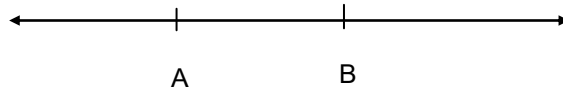
*Division I :
Introduction to Algebra*

Theme I : Foundation of Algebra : Numbers, Equations, and Graphs

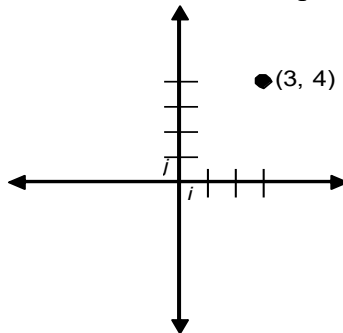
Lecon 1 : Numbers and Their Properties

Do You Know?

Numbers are the items that all aspects of mathematics utilize for effective communication. Quantitative and computation Algebra especially rely heavily on numbers. There are different kinds of numbers that one needs to know their properties. First, we remind one what a set is. **Set**, usually denoted by S or some capital letter, is a well-defined collection of objects. $A \subseteq B$ means that A is a **subset** of B ; that is all objects in A are also objects in B . **Natural numbers** or **counting numbers**, usually denoted by N , are the set of numbers consisting of $\{1, 2, 3, 4, \dots, N, \dots\}$. **Integers**, usually denoted by Z are the set of numbers consisting of $\{\dots-N\dots-3, -2, -1, 0, 3\dots N\dots\}$. **Rational numbers**, usually denoted by Q , are the set of numbers that can be expressed in the form p/q where p and q are integers and q is not zero $\{0\}$; they include all integers and all common fractions. **Irrational numbers**, usually denoted by \overline{Q} , are the set of numbers that can not be expressed as p/q where p and q are integers; they are numbers like $\sqrt{2}$, $2\sqrt{3+6}$, $5-\sqrt{7}$, etc. **Real numbers**, usually denoted by R , are the set of numbers that consist of all rational numbers and all irrational numbers; these numbers constitute all the numbers on a « line, » or the number line.



Complex numbers, usually denoted by C , is the set of the form $X + Y \cdot i$, where X and Y are real numbers and the Yi is called an imaginary part of the complex numbers; complex numbers represent all the numbers in the plane formed by the X and Y axis;



There are a few other types of numbers; however, they are not important for the study of algebra.

Numbers and Their Properties

Please remember that every real number is a complex number (where the Y is 0) ; every rational number is a real ; every irrational number is a real number ; every integer is a rational number ; every fraction is a rational number ; and every natural or counting number is an integer. In set theory notation $N \subseteq Z \subseteq Q \subseteq R \subseteq C$. In that most of algebra involves working with real numbers ; it is important to know the properties and rules (laws) for working with real numbers. The set of real numbers, R , has the following properties and follow the following rules (laws).

Some fundamental properties of Addition and Subtraction

A. **The Identity $\{0\}$ Property** (Addition and Subtraction) :

For all real numbers a , $a + 0 = 0 + a$; $a = a$

Example : $2\frac{3}{4} + 0 = 2\frac{3}{4} + 0 = 2\frac{3}{4}$

B. **Closure Property** (Addition and Subtraction) :

The sum (or difference) of two real numbers is a real number :

If x and y are real, then $x + y = z$ is a real number

C. **Associative Property** (Addition and Subtraction) :

If a , b , c are any three real numbers, then $a + (b + c) = (a + b) + c$

Example : $25 + (30\frac{1}{2} + 75) = (25 + 30\frac{1}{2}) + 75$

$$25 + 105\frac{1}{2} = 55\frac{1}{2} + 75$$

$$130\frac{1}{2} = 130\frac{1}{2}$$

D. **Commutative Property** (Addition and Subtraction) :

If a and b are any two real numbers, then $a + b = b + a$

Example : $125 + 375 = 375 + 125$

$$500 = 500$$

Some fundamental properties of Multiplication and Division

A. **The Identity Property $\{1\}$** (Multiplication and Division) :

For any real number a , $a \times 1 = 1 \times a = a$

Example : $525 \times 1 = 1 \times 525 = 525$

B. **Closure Property** : (Multiplication and Division)

For any real numbers a and b , $a \times b = c$, which is a real number.

Example : $100 \times 750 = 75,000$

Numbers and Their Properties

C. **Associative Property** (Multiplication and Division) :

If a, b and c are any real numbers, then $a \times (b \times c) = (a \times b) \times c$.

Example : $10 \times (5 \times 25) = (10 \times 5) \times 25$

$$10 \times 125 = 50 \times 25$$

$$1250 = 1250$$

D. **Commutative Property** (Multiplication and Division) :

If a and b are real numbers $a \times b = b \times a$

Example : $100 \times 30 = 30 \times 100$

$$3000 = 3000$$

E. **Distributive Property** (Multiplication and Division) :

If a, b, and c are real numbers, then $a \times (b + c) = (a \times b) + (a \times c)$

Example : $10 \times (10 + 60) = (10 \times 40) + (10 \times 60)$

$$10 \times 100 = 400 + 600$$

$$1000 = 1000$$

Operations And Their Procedure, What To Do First

If there is a statement with many operations for real numbers, there are some operations that we perform before performing others. In general we begin by performing operations within « interior » parentheses first, beginning with the innermost and continuing to the outermost. Whether within a single parenthesis or a stand-alone single statement, in general the operations to perform first are : exponents, multiplication, division, addition and subtraction.

Example :

$$\begin{aligned} & (3^2 \times 2 - 12/4 + (2^4 \times 10 + 2)) \\ &= (3^2 \times 2 - 12/4 + (8 \times 10 + 2)) \\ &= (3^2 \times 2 - 12/4 + (80 + 2)) \\ &= (3^2 \times 2 - 12/4 + 82) \\ &= (9 \times 2 - 12/4 + 82) \\ &= (18 - 3 + 82) \\ &= (100 - 3) \\ &= 97 \end{aligned}$$

Multiplying and Dividing by Zero {0}

For any number a, $a \times 0 = 0$

For any number a, $a/0$ is undefined. We are not permitted to divide by 0.

Numbers and Their Properties

Numbers and Fractions

A rational number is a number that can be expressed as p/q where p and q are integers ; the p is called the numerator and the q is called the denominator, every rational number is a fraction.

Example

Type of Fraction

$$-\frac{6}{10}$$

proper (numerator smaller than denominator)

$$\frac{7}{25}$$

proper

$$\frac{15}{4}$$

improper (numerator is larger than denominator)

For any integer z , $\frac{z}{1} = z$ integer

$$9\frac{7}{8}$$

Integer and fraction, mixed

$$\left(\frac{1}{2}\right) / \left(\frac{7}{5}\right)$$

Compound (numerator and denominator are each fraction)

For any integer a , $\frac{a}{a} = 1$ integer, multiplicative identity.

Operations on Fractions

Reciprocal Fractions : If a/b is a fraction then b/a is its reciprocal fraction

Example : $\frac{9}{25}$ has $\frac{25}{9}$ as its reciprocal.

Equivalent Fractions : If a/b is any fraction, $\frac{a}{b} \times \frac{c}{c} = \frac{ac}{bc}$ is a fraction of the same value (measure). Why ?

Examples : $\frac{3}{4} \times \frac{110}{10} = \frac{30}{40}$

$$\frac{1}{2} \times \frac{13}{13} = \frac{13}{26}$$

Numbers and Their Properties

Multiplication of Fractions

If a/b and c/d are fractions, then we multiply these fractions by multiplying numerator times numerator and denominator times denominator.

Example : $\frac{3}{4} \times \frac{2}{5} = \frac{6}{20}$

$$\frac{3}{7} \times \frac{1}{10} = \frac{3}{70}$$

$$\frac{7}{16} \times \frac{2}{3} = \frac{14}{48}$$

Addition of Fractions

To add a/b and c/d both fractions have to be changed to two equivalent fractions with the same denominator, then the two numerators are added.

Example : Add $\frac{3}{4} + \frac{2}{5}$

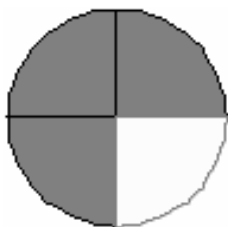
$$\frac{3}{8} + \frac{2}{8} = \frac{3+2}{8} = \frac{5}{8}$$

$$\frac{3}{5} + \frac{1}{5} = \frac{3+1}{5} = \frac{4}{5}$$

To add two fractions if they don't have the same denominator, you first need to convert them so that they both do have the same denominator. To do this, use the fact that you can multiply both the top and the bottom by the same number and still keep the same value. For example,

$$\frac{1}{2} + \frac{1}{4} = \frac{1 \times 2}{2 \times 2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{2+1}{4} = \frac{3}{4}$$

$$\frac{1}{2} + \frac{1}{3} = \frac{1 \times 3}{2 \times 3} + \frac{1 \times 2}{3 \times 2} = \frac{3}{6} + \frac{2}{6} = \frac{3+2}{6} = \frac{5}{6}$$



$$\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$$



$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

Numbers and Their Properties

Reciprocals and Division of Fractions

The reciprocal of a fraction is found simply by turning the fraction upside down (in other words, putting the denominator on top and the numerator on the bottom). For example,

The reciprocal of $5/3$ is $3/5$.

To divide two fractions, multiply the first fraction by the reciprocal of the second fraction:

$$2/3 \div 1/4 = 2/3 \times 4/1 = 8/3$$

Summary of Rules for Addition, Subtraction and Division of Fractions

Multiplication $a/b \times c/d = ac/bd$

Addition when the denominators are the same $a/b + c/b = \frac{a + c}{b}$

Addition when the denominators are different $a/b + c/d = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$

Subtraction $a/b - c/d = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}$

Simplification of compound fractions (division) $\frac{a/b}{c/d} = \frac{ad}{bc}$

In all rules for fractions, it is automatically assumed that no denominators are equal to zero.

Decimal Fractions

Two fractions with different denominators are hard to compare. For example, at first glance it is difficult to tell whether or not $9/16$ is greater than $25/44$. Therefore, it often helps to convert fractions into equivalent fractions with the same denominator. The most convenient denominators to use for this purpose are the numbers 10, 100, 1,000, and so on. For example,

$$\frac{9}{6} - \frac{3}{18} = \frac{9}{6} - \frac{1}{6} = \frac{9-1}{6} = \frac{8}{6} = \frac{4}{3}$$

Numbers and Their Properties

To save writing in these situations, we will not write out the number in the denominator. Instead, we will put a period (.), called a decimal point, in front of the numerator. (Often, a zero is placed before the decimal point so that the decimal point will not be overlooked.) For example,

$$\frac{1}{2} = 0.5 \qquad \frac{1}{4} = 0.25 \qquad \frac{3}{5} = 0.6$$

If only one digit is written to the right of the decimal point, then the denominator is 10.

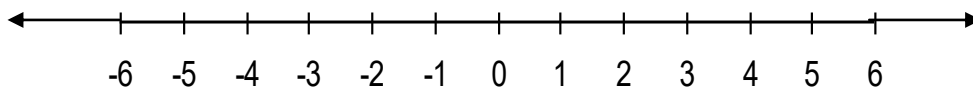
$$\begin{array}{lll} 0.1 = 1/10 & 0.2 = 2/10 & 0.3 = 3/10 \\ 0.4 = 4/10 & 0.5 = 5/10 & 0.9 = 9/10 \end{array}$$

If there are two digits to the right of the decimal point, then the denominator is 100.

$$0.15 = 15/100 \qquad 0.33 = 33/100 \qquad 0.75 = 75/100$$

Activity 1

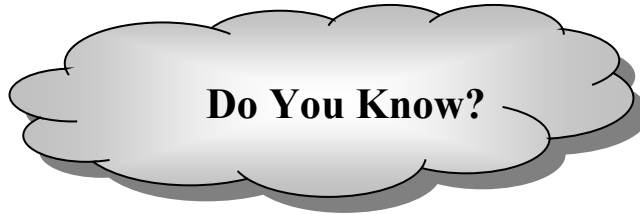
1. Locate these numbers on the number line $\{5, -3, 9/2, -1.5, -17/4, 8/3\}$



2. What number A is $9/2 + 8/3$?
3. What number B is $9/2 - 8/3$?
4. What number C is $(9/2)(-1.5)$?
5. What is D is $5 / (8/3)$?
6. What is the additive inverse of (-3) ?
7. What is the multiplicative inverse of $8/3$?
8. What is $[(-17/4) + 9/2 + 8/3]$?
9. $7 + 4i$ is what type of number ?
10. Name four different types of numbers that 25 represents.

Theme I : Foundation of Algebra : Numbers, Equations, and Graphs

Lesson 2: Equations and Inequalities



An equation is a number sentence that uses the equal sign to show that two amounts are equal.

Here are examples of equations and their solution :

Equation	Solution
$10y - 20 = 0$	$y = 2$
$6y + 4 = 8y - 10$	$y = 7$
$130y + 6y - 20 = 68y + 184$	$y = 3$
$46y + 38y + 10 = 54y + 10$	$y = 0$

Equation	Solution
$3x - 6 = 12$	$x = 6$
$3x - 12 = 12$	$x = 8$
$4x - 4 = 12$	$x = 4$
$4x - 20 = 12$	$x = 8$
$5x - 18 = 12$	$x = 6$

Equations and Inequalities

Equations

When an equation contains one unknown quantity, then it is often possible to solve for the value of the unknown that makes the equation true. This can be done by using the golden rule of equations (that is, perform exactly the same operation on each side of the equation) until the unknown has been isolated on one side of the equation. (However, some types of equations are too complicated to solve in this manner.)

When a single equation contains two or more unknown quantities, then it is often possible to find many possible solutions for the equation. In this case a solution consists of a set of values for each of the unknowns in the equation. For example, if the two unknowns are x and y then a solution to the equation consists of two values: one for x and one for y . (If the number of equations is the same as the number of unknowns that appear, then it is often possible to find a single solution that makes all of the equations true simultaneously.)

- Some special equations are true for all possible values of the unknowns they contain. Equations of this kind are called identities. Here are some examples of identities.

$$4x = x + x + x + x$$

$$3(a + b) = 3a + 3b$$

$$(a + b)(c + d) = ac + ad + bc + bd$$

- Sometimes an equal sign with three bars (\equiv) is used in place of the regular equal sign to indicate that an equation is an identity.
- An equation of the form $ax + b = 0$, in which x represents an unknown number and a and b represent known numbers, is said to be a linear equation. The reason for this name will become clear later, when we will find that a linear equation can be represented as a line on a graph.
- A statement of the form “ x is less than y ,” written $x < y$, or “ a is greater than b ,” written $a > b$, is called an inequality. The arrow in the inequality sign always points to the smaller number. Inequalities containing only numbers will either be true (for example, $10 > 7$) or false (for example, $4 < 3$). Inequalities containing variables (such as $x < 3$) will usually be true for some values of the variable but not for others.

Equations and Inequalities

GOLDEN RULE OF EQUATIONS

Whatever you do unto one side of an equation, do the same thing unto the other side of the equation. Then the equation will remain true (assuming that it was true to begin with). In particular, you can

- Add the same number to both sides,
- Subtract the same number from both sides,
- Multiply both sides by the same number,
- Divide both sides by the same number.

However, you cannot divide both sides of an equation by zero, since (as we later found out) you cannot divide anything by zero. Also, although strictly speaking it is legal, it gets you nowhere to multiply both sides of an equation by zero.

If you start with a false equation and do the same thing to both sides of the equation, then of course you will still have a false equation.

Implied Multiplication Rule

When two letters, or a letter and a number, are written adjacent to each other with no operation symbol between them, then it is implied that they are meant to be multiplied together.

The equation we were trying to solve was

$$10 \times x + 5 \times x - 35 = x + 7$$

Using implied multiplication, we rewrote the equation:

$$10x + 5x - 35 = x + 7$$

Subtracting x from both sides:

$$10x + 5x - x - 35 = 7$$

Adding 35 to both sides:

$$10x + 5x - x = 7 + 35$$

Simplifying:

$$14x = 42$$

Then we found the final solution: $x = 3$

Equations and Inequalities

Worksheet: Equations

Solve these equations for x:

1. $x + 32 = 74$; $x =$
2. $x - 12 = 32$; $x =$
3. $8x = 48$; $x =$
4. $x \div 3 = 15$; $x =$
5. $2x + 3 = 21$; $x =$
6. $4x - 10 = 54$; $x =$
7. $12x - 8 = 3x + 64$; $x =$
8. $100 - 4x = 20 + 6x$; $x =$

Worksheet Answers: Equations

1. $x + 32 = 74 - 32 = 42$
2. $x - 12 = 32$; $x = 32 + 12 = 44$
3. $8x = 48$; $x = 48 \div 8 = 6$
4. $x \div 3 = 15$; $x + 3 \times 15 = 45$
5. $2x + 3 = 21$; $2x = 21 - 3$; $x = 18 \div 2 = 9$
6. $4x - 10 = 54$; $4x = 54 + 10$; $x = 64 \div 4 = 16$
7. $12x - 8 = 3x + 64$; $12x - 3x = 64 + 8$; $9x = 72$; $x = 8$
8. $100 - 4x = 20 + 6x$; $100 - 20 = 6x + 4x$; $80 = 10x$; $x = 8$

Equations and Inequalities

Activit 1

Solve these equations for x

1. $3x + 16 = 22$
 2. $14 - x = 10$
 3. $34 - 10x = 6x + 2$
 4. $7x + 5 = 75$
 5. $3x - 8 = 16$
 6. $6 - 7x + 20 = 5$
 7. $9 - x = 1$
 8. $x + 2x + 3x + 4x + 5x = 45$
 9. $100 - 4x = 60$
 10. $10 + 5x = 110$
 11. $10 + 5x = 100 - 4x$
 12. $12 + ax = 16$
 13. $b + ax = 12$
 14. $b + ax = c$
 15. $ax = bx + c$
 16. $ax - c = d$
 17. $bx = h$
 18. $az - b = cx + d$
19. If you travel at 45 miles per hour for 5 hours, how far will you travel?
 20. If you travel at 30 miles per hour, how long will it take you to travel 3,000 miles?
 21. Two people start driving toward each other, starting from two towns 100 miles apart. If one person travels 30 miles per hour and the other person travels 20 miles per hour, how long will it take until they meet?
 22. Answer the same question as in the previous problem, only this time assume that the first person travels v_1 miles per hour and the second person travels v_2 miles per hour.
 23. Suppose that you are trying to earn \$48. You have your choice between working at a hard job that pays \$12 per hour, or at an easy job that pays \$6 per hour, or you can work some time at both jobs. List some possible ways that you can earn the \$48.
 24. If CDs cost \$12, pizzas cost \$6, and you have \$60 to spend on these two goods this month, list the possible numbers of pizzas and CDs that you can buy.
 25. Farmer Floran has roosters and horses. The animals have a total of 88 feet and 40 wings. How many horses and how many roosters are there?
 26. If the city plans to plant a total of 18 trees on two streets, putting twice as many trees on Elm Street as on Maple Street, how many trees will be planted on each street?
 27. If Chapter 1 in a book contains five more pages than does Chapter 2, and there are 65 total pages in both chapters, how many pages does each chapter contain?
 28. What two consecutive numbers add up to 63?
 29. What three consecutive numbers add up to 75?
 30. T.J. can type twice as fast as J.R. If they both spend an equal amount of time on a 600-page manuscript, how many pages will they each have typed?
 31. A plane is traveling between two cities that are 660 miles apart. The plane can travel at 200 miles per hour relative to the air, but it will go faster relative to the ground if it is helped by the wind. If the plane completes the trip in 3 hours, how fast was the wind blowing?
 32. You have 90 feet of fencing. You will build a rectangular enclosure whose length is twice the width. What are the dimensions of the enclosure?

Equations and Inequalities

Activity 1 [Cont.'d]

\

Postulates of Equality

- Any number is equal to itself. In symbols, if a is any number, then

$$a = a$$

(The technical name for this property is the reflexive property of equality.)

- You can reverse the two sides of an equation whenever you feel like it. (Technical name: symmetric property of equality.) In symbols, if a and b are any two numbers,

$$a = b \quad \text{means the same thing as} \quad b = a$$

- If two numbers are both equal to a third number, they must be equal to each other. (Technical name: transitive property.) In symbols, if a , b , and c are any three numbers, and

$$\text{If } a = c \text{ and } b = c, \text{ then } a = b$$

- Golden rule of equations: Whatever you do to one side of an equation, do exactly the same thing to the other side. In symbols, let a , b , and c be any three numbers. If $a = b$, then

$$a + c = b + c$$

$$a - c = b - c$$

$$a \times c = b \times c$$

$$\frac{a}{c} = \frac{b}{c}$$

$$c \neq 0$$

(In the last equation, c must not be zero.)

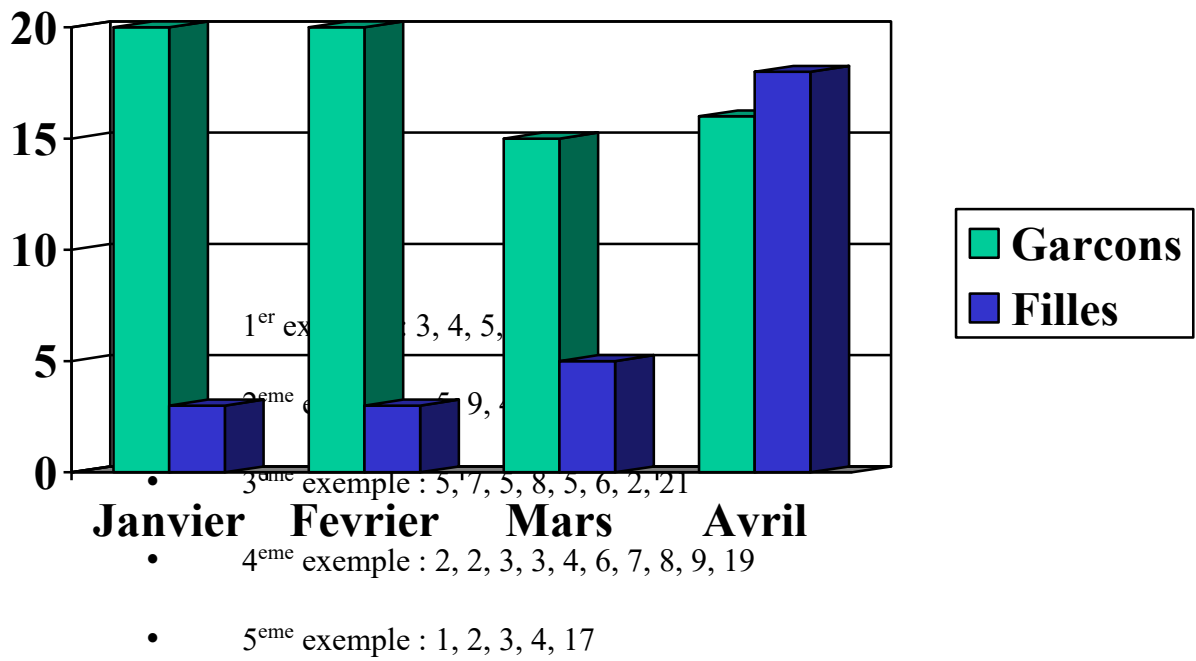
- Substitution property: If $a = b$, then you can substitute a in the place of b anywhere that b appears in an expression.

Equations and Inequalities

Activity 2 [Cont.'d]

<p>Do You Know How?</p> <p>Inequalities on a Number Line</p> <p>Name three solutions to each inequality and graph all the solutions on a number line.</p> <p>1. $m < 10$ 2. $z > 4$ 3. $k > 14$ 4. $p < 5$</p>	<p>Do You Understand?</p> <p>A. Tell how you know a number is a solution to an inequality.</p> <p>B. Explain how your number line in Exercise 2 shows that 7 is a solution of $z > 4$.</p>
<p>Translating Words to Equations</p> <p>Write an equation for each sentence.</p> <p>5. 5 more than s is equal to 14. 6. t less than 16 is 13. 7. k times 8 is 24.</p>	<p>C. Tell how you wrote the equation in Exercise 7.</p> <p>D. Explain how you know which operation to use in Exercise 6.</p>
<p>Equations and Graphs</p> <p>Graph each equation on a separate coordinate grid.</p> <p>8. $y = x + 3$ 9. $7 = 3x$</p> <p>Use the equation $y = 5x + 3$. Find the value of y for each value of x.</p> <p>10. $x = 2$ 11. $x = 9$</p>	<p>E. Tell how you graphed the equation in Exercise 8.</p> <p>F. Name 5 ordered pairs on the graph of the equation in Exercise 9.</p>

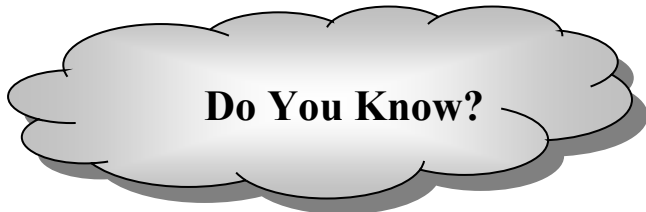
Ce diagramme à bâtons ou barres présente des données. La longueur d'une barre représente une quantité, une valeur (le nombre de filles ou de garçons né(e)s durant ce mois).



Theme I : Foundation of Algebra :

Numbers, Equations, and Graphs

Lesson 3 : Exponents, Powers, Roots and Radicals



Exponential notation is shorthand for repeated multiplication:

$$3 \cdot 3 = 3^2 \text{ and } (-2y) \cdot (-2y) \cdot (-2y) = (-2y)^3$$

In the notation a^n , a is the base, and n is the exponent. The whole expression is « a to the n th power, » or the « n th power of a , or, simply, « a to the n . »

a^2 is « a squared ; » a^3 is « a cubed. » $(-a)^n$ is not necessarily the same as $-(a^n)$.

Ex. $(-4)^2 = 16$, whereas $-(4^2) = -16$.

Following the order of operation rules, $-a^n = -(a^n)$.

A^n means n a 's multiplied together. For example,

$$11^2 = 11 \times 11 = 121$$

$$5^4 = 5 \times 5 \times 5 \times 5 = 625$$

$$10^3 = 10 \times 10 \times 10 = 1,000$$

$$2^{10} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 1,024$$

$$6^1 = 6$$

$$6^0 = 1 ; a^0 = 1 \text{ for any number } a$$

« We can write a squared with exponents: a squared = a^2 »

X	x^2	x^3	x^4	x^5	x^6	x^7
1	1	1	1	1	1	1
2	4	8	16	32	64	128
3	9	27	81	243	729	2,187
4	16	64	256	1,024	4,096	16,384
5	25	125	625	3,125	15,625	78,125
6	36	216	1,296	7,776	46,656	279,936
7	49	343	3,401	16,807	117,649	823,543
8	64	512	4,096	32,768	262,144	2,097,152
9	81	729	6,561	59,049	531,441	4,782,969
10	100	1,000	10,000	100,000	1,000,000	10,000,000

Exponents, Powers, Roots and Radicals

Consider:
$$\frac{5^7}{5^3} = \frac{5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5 \times 5}$$

Cancel three 5's from the top and bottom:

$$\begin{aligned} &= 5 \times 5 \times 5 \times 5 \\ &= 5^4 \end{aligned}$$

Some more examples convinced us that in general

$$x^a/x^b = x^{a-b}$$

We also found that $(x^a)^b = x^{ab}$. For example,

$$\begin{aligned} (3^2)^4 &= (3^2) \times (3^2) \times (3^2) \times (3^2) \\ &= 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \\ &= 3^8 \end{aligned}$$

The final rule we found was that $(xy)^a = x^a y^a$. For example,

$$\begin{aligned} (5 \times 3)^2 &= (5 \times 3) \times (5 \times 3) \\ &= (5 \times 5) \times (3 \times 3) \\ &= 5^2 \times 3^2 \\ &= 225 \end{aligned}$$

« We have pinned down the behavior of exponents pretty well, » Recordis said. « They will have no choice but to obey these four laws. »

Exponents, Powers, Roots and Radicals

Rules of Exponents

Product of power: $a^m a^n = a^{m+n}$

If the bases are the same, then to multiply, simply add their exponents Ex: $2^3 \cdot 2^8 = 2^{11}$.

Quotient of powers: $\frac{a^m}{a^n} = a^{m-n}$

If the bases of two powers are the same, then to divide, subtract their exponents.

Exponentiation powers: $(a^m)^n = a^{mn}$

To raise a power to a power, multiply exponents.

Power of a product: $(ab)^n = a^n b^n$

Quotient of a product: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Exponentiation distributes over multiplication and division, but not over addition or subtraction. Ex: $(2xy)^2 = 4x^2y^2$, but $(2 + x + y)^2 \neq 4 + x^2 + y^2$

Zeroth power: $a^0 = 1$

To be consistent with all the other exponent rules, we set $a^0 = 1$ unless $a = 0$. The expression 0^0 is undefined.

Negative powers: $a^{-n} = \frac{1}{a^n}$

We define negative powers as reciprocals of positive powers. This works well with all other rules. Ex: $2^3 \cdot 2^{-3} = \frac{2^3}{2^3} = 1$.

Also, $2^3 \cdot 2^{-3} = 2^{3+(-3)} = 2^0 = 1$.

Fractional powers: $a^{\frac{1}{n}} = \sqrt[n]{a}$

This definition, too, works well with all other rules.

Exponents, Powers, Roots and Radicals

Activity 1

Simplify these expressions. Express all of the negative powers as fractions.

1. $3^5/3^2$
2. $2^{10}/2^6$
3. $5/5^2$
4. $(1.16)^2/(1.16)^4$
5. $(3.45)^3/(3.45)^{-2}$
6. x^{-3}/y^{-3}
7. $3r^3/4r^2$
8. $10a^2b^3c/abc$
9. $16a^3b^4c/8a^3b^6c^2$
10. r^2/r
11. $1/x^{-1}$
12. b^{-3}
13. $a^{-2}c^2$
14. $1/(x^{-1} + x^{-2})$
15. $3^7 \cdot 3^{-5}$
16. $[(-4)^3]$
17. $(2 \cdot 3)^3$
18. $4^{12}/4^7$
19. $8 \frac{3}{8^3}$
20. $(xy^2)^3$
21. $[(a^2b)/(ab^3)]^3$
22. $(a^{-2})^3/(a^{-3})^2$
23. $(5^3 \cdot 5^{-2})/5^4$
24. $[(5ab^{-2})/(7a^{-2}b)]^2$
25. $(2x^2y^3z^{-2})/(x^5y^{-1}z^4)$

Exponents, Powers, Roots and Radicals

Activity 2

Simplify these radicals if possible. For example, $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \sqrt{2} = 3\sqrt{2}$.

- | | | | |
|-----|-----------------|-----|---------------------------|
| 1. | $\sqrt{12}$ | 15. | $\sqrt{24a^2x}$ |
| 2. | $\sqrt{100}$ | 16. | $\sqrt{75a^3x}$ |
| 3. | $\sqrt{9/16}$ | 17. | $\sqrt[10]{1,024}$ |
| 4. | $\sqrt{8}$ | 18. | $\sqrt[4]{81}$ |
| 5. | $\sqrt{72}$ | 19. | $\sqrt[5]{100,000}$ |
| 6. | $\sqrt{32}$ | 20. | $\sqrt[9]{1}$ |
| 7. | $\sqrt{44}$ | 21. | $\sqrt[4]{1/256}$ |
| 8. | $\sqrt{a^4x^2}$ | 22. | $\sqrt[4]{4}$ |
| 9. | $\sqrt{4x^6}$ | 23. | $\sqrt[4]{9}$ |
| 10. | $\sqrt{12y^2}$ | 24. | $\sqrt[10]{32}$ |
| 11. | $\sqrt{4a^2}$ | 25. | $\sqrt[3]{729}$ |
| 12. | $\sqrt{9x}$ | 26. | $\sqrt[6]{a^3b^6c^{12}}$ |
| 13. | $\sqrt{48x}$ | 27. | $\sqrt[4]{a^{12}/b^{24}}$ |
| 14. | $\sqrt{18x}$ | | |

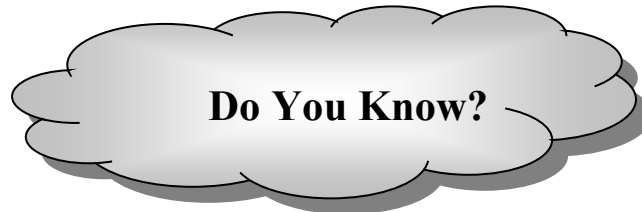
Remove the radicals from the denominators in these expressions.

- | | | | |
|-----|--------------------|-----|---------------------------------|
| 28. | $\sqrt[3]{5}$ | 32. | $(1 + \sqrt{2})/(1 - \sqrt{2})$ |
| 29. | $1/(\sqrt{5} + 4)$ | 33. | $(1 - \sqrt{5})/(1 - \sqrt{2})$ |
| 30. | $10/\sqrt{5}$ | 34. | $1/(\sqrt{2} + \sqrt{8})$ |
| 31. | $5/(\sqrt{6} + 9)$ | 35. | $5/(\sqrt{6} + \sqrt{54})$ |

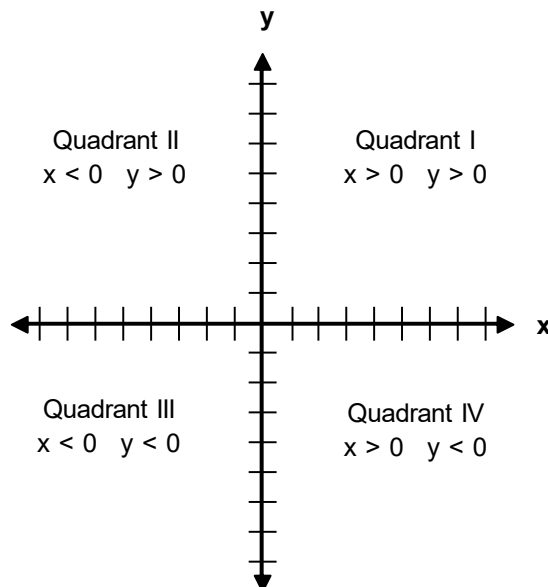
Theme I : Foundation of Algebra :

Numbers, Equations, and Graphs

Lesson 4: Graphing Lines in the Cartesian Plane



A flat surface extending off to infinity is called a plane. The x-axis and the y-axis divide a plane into four regions called quadrants. The quadrant where both x and y are positive is called quadrant I. The other three quadrants are labeled below.



The system of x- and y-coordinates identified here is called the rectangular or Cartesian coordinate system. (The Cartesian system is named after the philosopher and mathematician Descartes.) There are other types of coordinate systems.

Graphing Lines in the Cartesian Plane

Graphing lines in the Cartesian Plane

Terminology of the Cartesian Plane

x-axis: Usually, the horizontal axis of the coordinate plane. Positive distances are measured to the right; negative, to the left.

y-axis: Usually, the vertical axis of the coordinate plane. Positive distances are measured up; negative, down.

Origin: (0,0), the point of intersection of the x-axis and the y-axis.

Quadrants: The four regions of the plane cut by the two axes. By convention, they are numbered counterclockwise starting with the upper right.

Point: A location on a plane identified by an ordered pair of coordinates enclosed in parentheses. The first coordinate is measured along the x-axis; the second, along the y-axis. Ex: The point (1,2) is 1 unit to the right and 2 units up from the origin. Occasionally (rarely), the first coordinate is called the abscissa; the second, the ordinate.

Lines in the Cartesian Plane

A straight line is uniquely identified by any two points, or by any one point and the incline, or slope, of the line.

Slope of a line: The slope of a line in the Cartesian plane measures how steep it is.

Graphing Linear Equations

A linear equation in two variables (say x and y) can be manipulated – after all the x-terms and y-terms and constant terms have been grouped together – into the form $Ax + By = C$. The graph of the equation is a straight line (hence the name).

Using the slope to graph: plot one point of the line. If the slope is expressed as a ratio of small whole numbers $\frac{r}{s}$, keep plotting points r up and s over from the previous point until you have enough to draw the line.

Finding intercepts: To find the y-intercept, set $x=0$ and solve for y. To find the x-intercept, set $y=0$ and solve for x.

Slope-intercept form: $y = mx + b$

Graphing Lines in the Cartesian Plane

One of the easiest-to-graph forms of a linear equation.

m is the slope.

B is the y -intercept, $(0,b)$ is a point on the line.

Point-Slope Form: $y - k = m(x - h)$

m is the slope.

(h, k) is a point on the line.

Standard Form: $Ax + By = C$

Less thinking: Solve for y and put the equation in to slope-intercept form.

Less work: Find the x - and the y -intercepts. Plot them and connect the line.

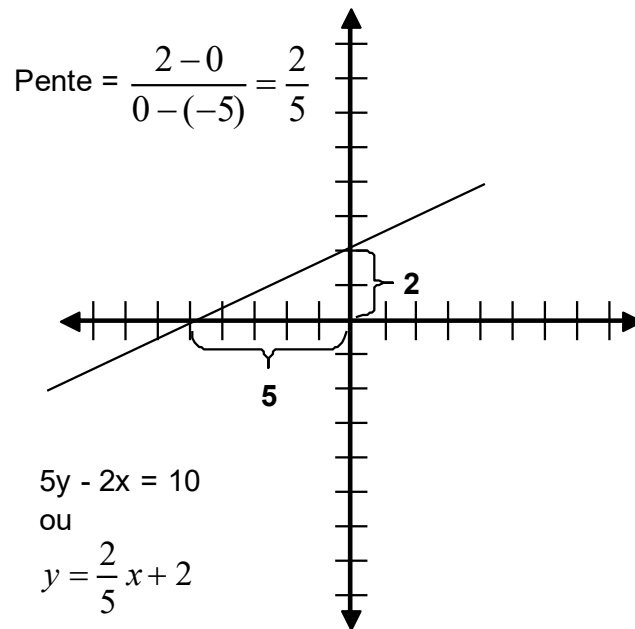
Slope: $-\frac{A}{B}$.

y -intercept: $\frac{C}{B}$

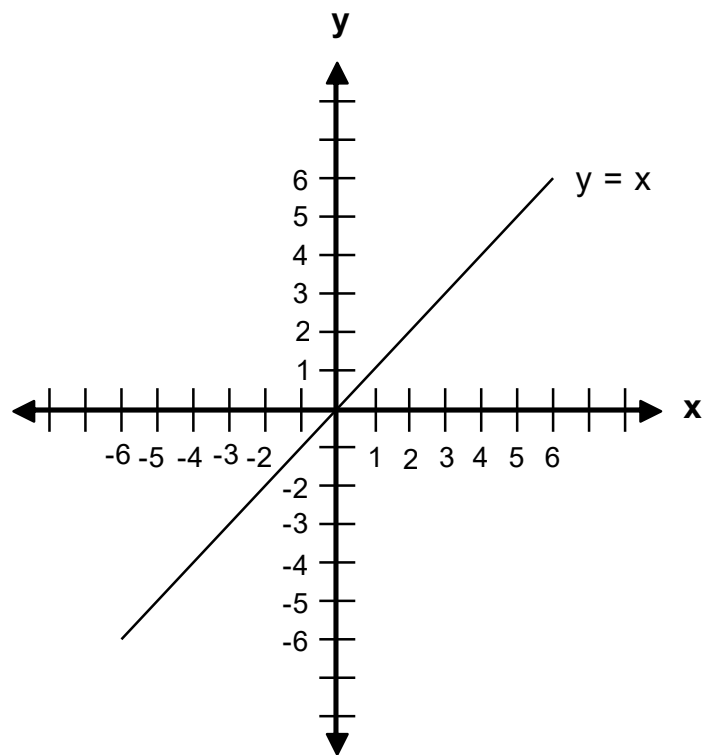
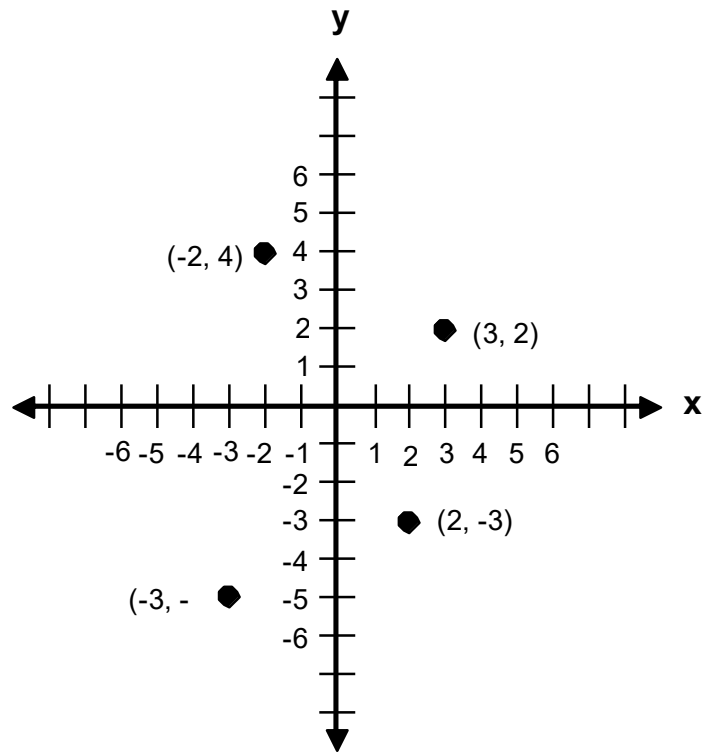
x -intercept: $\frac{C}{A}$

Any other form

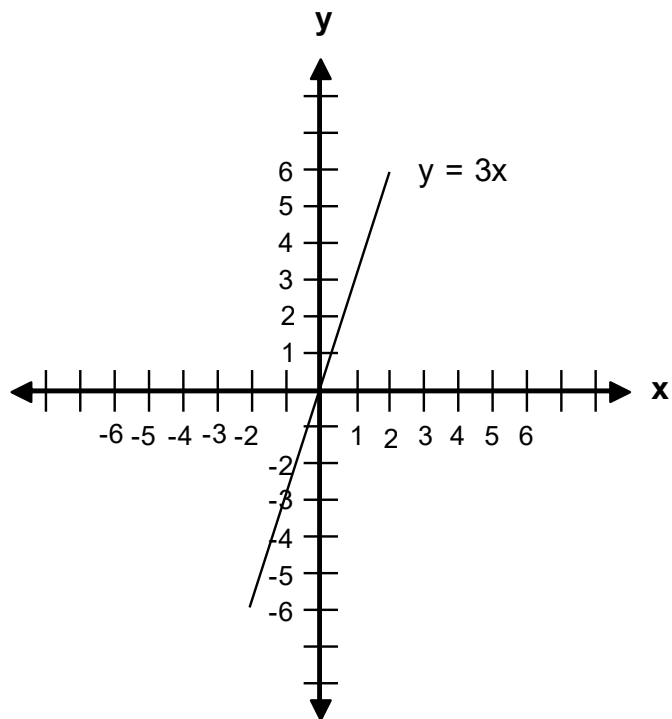
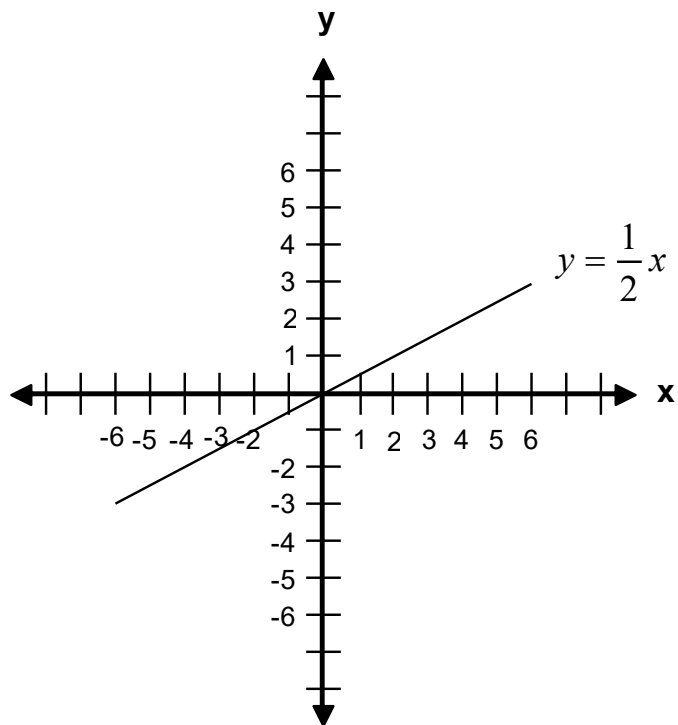
If you are sure that an equation is linear, but it isn't in a nice form, find a couple of solutions. Plot those points. Connect them with a straight line. Done.



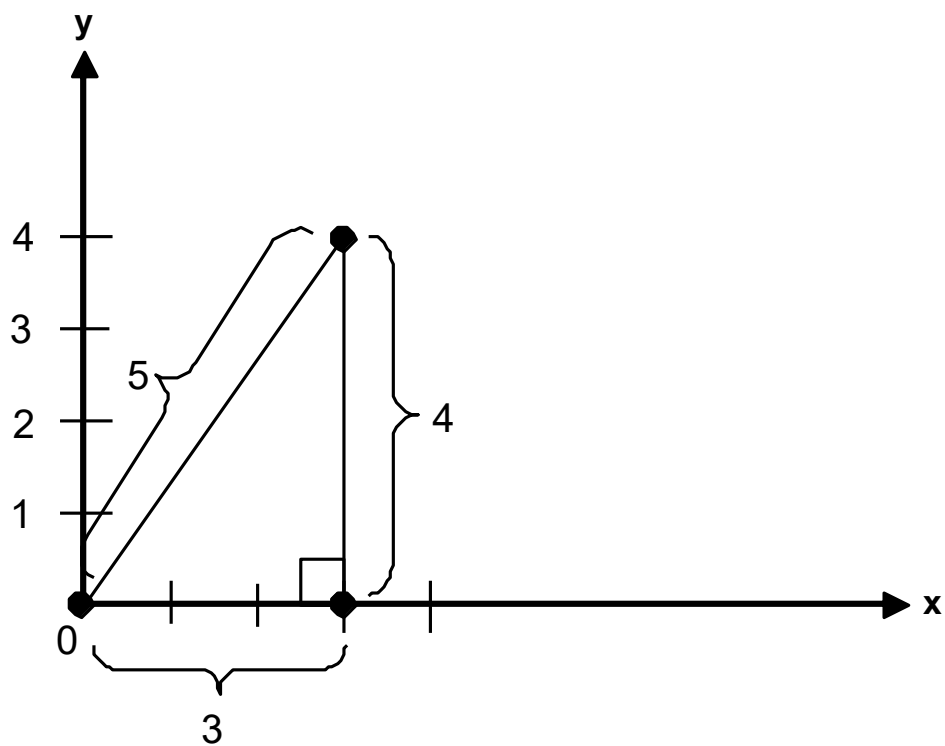
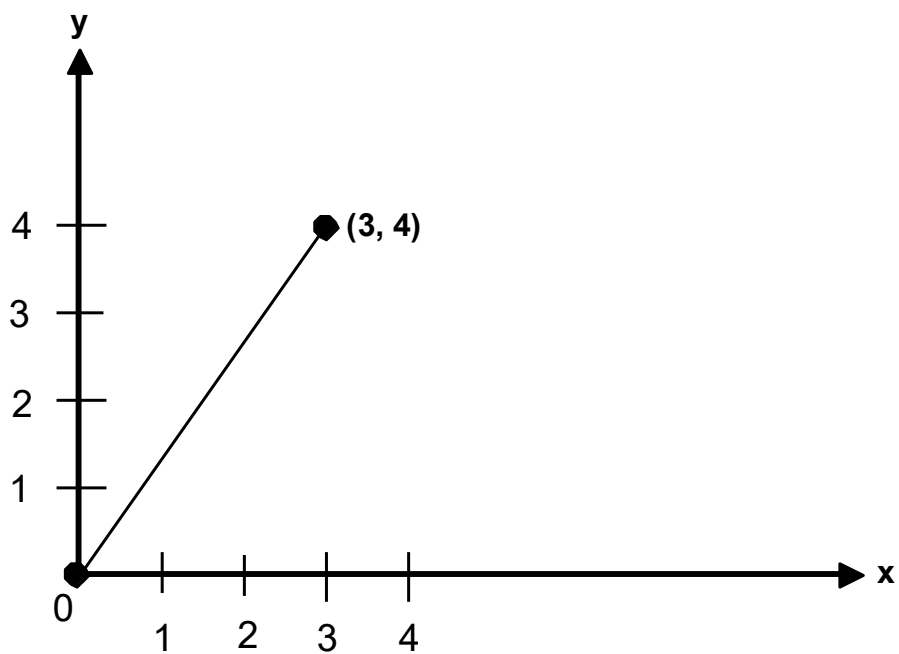
Graphing Lines in the Cartesian Plane



Graphing Lines in the Cartesian Plane



Graphing Lines in the Cartesian Plane



Graphing Lines in the Cartesian Plane

A general formula for the distance from the origin to the point (x, y)

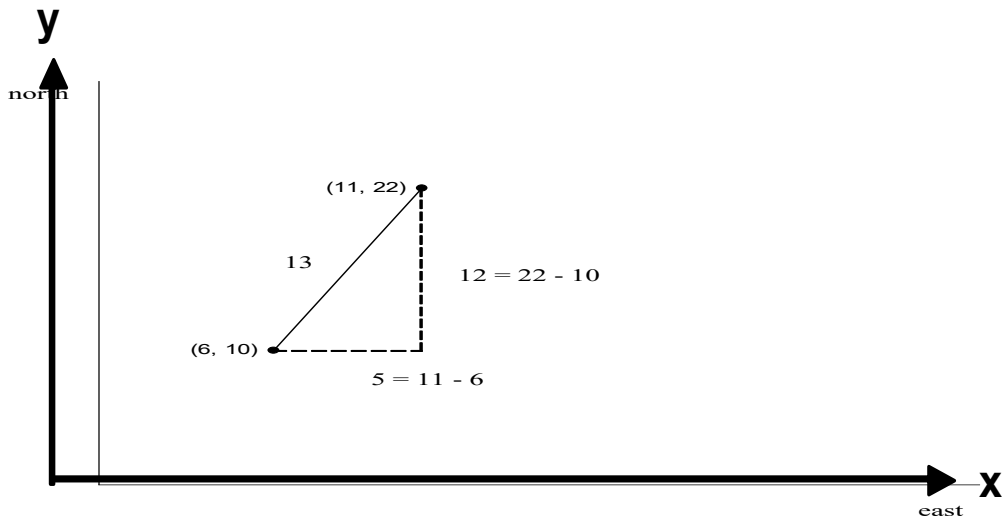
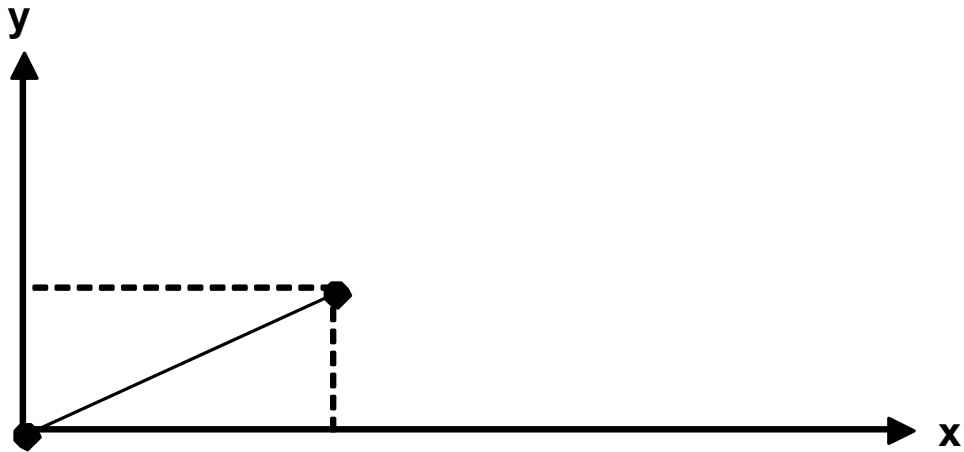
$$d = \sqrt{x^2 + y^2}$$

The distance between two points will be

$$d = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

In general, the distance between the two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



Graphing Lines in the Cartesian Plane

Activity 1

Find the distance between the following pairs of points. Then find the equation of the line connecting the points.

1. (0, 0) and (6, 8)
2. (2, 3) and (6, 10)
3. (-1, 0) and (1, 1)
4. (-4, 6) and (2, 10)
5. (9, 11) and (8, 17)
6. (16, 4) and (0, -3)
7. (2, 4) and (2, 12)
8. (3, 10) and (10, 10)

Find the coordinates of the points where the following lines cross the x- and y-axes.

9. $5y - 10x = 0$
10. $x + y = 1$
11. $2x + y = 10$
12. $100x - y = 1,000$
13. $\frac{1}{4}x + \frac{1}{3}y = 1$
14. $0.37x - 4.12y = .34$

Find the equations of lines that pass through the given points and have the indicated slopes.

15. slope: $\frac{1}{2}$ point: (6, 4)
16. slope: 0 point: (5, 10)
17. slope: m point: (1, 2)
18. slope: m point: (a, 5)
19. slope: 100 point: (1,025, 1,030)
20. slope: m point: (-5, -6)
21. $x + y \leq 30; x \geq 0; y \geq 0$
22. $4x + 9y \leq 72; x \geq 0; y \geq 0$
23. $10x + 5y \leq 120; x \geq 0; y \geq 0$

Theme I : Foundation of Algebra : Numbers, Equations, and Graphs

Lesson 5: Function Notation

Do You Know?

Function Notation

A function is a rule that converts one number to another number. If the value of y (called the dependent variable) depends on the value of x (called the independent variable), then we can write the equation

$$y = f(x)$$

(read as y equals f of x). In this expression f stands for function. Think of the function as a machine: Insert the value of x into the input slot, and the value of y comes out of the output slot.

In a particular circumstance you need to specify which function you mean by listing a formula; for example, $f(x) = 4x^2$ defines a function. Then the result (or output) of the function depends on the specific number (or expression) that is used as the argument (or input) to the function. For this example, $f(3) = 4 \times 3^2 = 36$; $f(12) = 4 \times 12^2 = 576$, $f(5a)^2 = 4 \times 25a^2 = 100a^2$, and $f(ab) = 4(ab)^2$.

There is a unique value of $f(x)$ determined by each value of x . In other words, if $x = 10$, then you can determine unambiguously the value of $f(10)$ for any specific function f . However, different values of x might give the same result for $f(x)$. For example, if $f(x) = x^2$, then $f(10)$ and $f(-10)$ have the same value (they both equal 100). In the extreme case where $f(x)$ equals a constant value, such as $f(x) = 17$, then all values of x lead to the same value of $f(x)$.

Sometimes two functions are chained together so that the output of one function becomes the input of another function. For example, if $g(y) = \sqrt{y}$ and $h(x) = 2x$, then we can create the function $f(x) = g(h(x)) = \sqrt{2x}$.

If $y = f(x)$, and $x = g(y)$ for all values of x and y , then g is the inverse function of f . It has the effect of reversing the action of f and returning you to your original number. If f and g are chained together, then $f(g(x)) = x$.

The set of all possible values for the input to the function is called domain; the set of all possible values for the output is called the range.

Function Notation

Graphing the function

Typically the independent variable (called x unless there is a more appropriate name for a particular situation) is measured along the horizontal axis, and the dependent variable (typically called y) is measured along the vertical axis. For a particular value x_1 , the height of the curve (or line) at that point gives the value of $y = f(x_1)$. Obviously the graph cannot show all possible real number values of x , but if the graph shows all the values of x that you are interested in, you can read off the value of y directly from the graph (although reading values from the graph is usually less accurate than calculating values from the formula, if a formula is available).

Function	Domain	Range
$f(x) = 3x$	real numbers	Real numbers
$f(x) = 1/x$	real numbers (except 0)	Real numbers (except 0)
$f(x) = x^2$	real numbers	$f(x) \geq 0$
$f(x) = \sqrt{x}$	$x \geq 0$	$f(x) \geq 0$
$f(x) = x $	real numbers	$f(x) \geq 0$
$f(x) = \sqrt{1-x^2}$	$-1 \leq x \leq 1$	$0 \leq y \leq 1$

Equations for Lines

An equation of the form

$$y = mx + b$$

can be represented by a line with slope = m , y -intercept = b , and x -intercept = $-b/m$.

An equation of the form

$$ax + by = c$$

can be represented as a line with slope = $-1/b$, y -intercept = c/b , and x -intercept = c/a .

An inequality of the form

$$ax + by \leq c$$

or

$$ax + by \geq c$$

can be represented as an area on a graph. First draw the line

$$ax + by = c$$

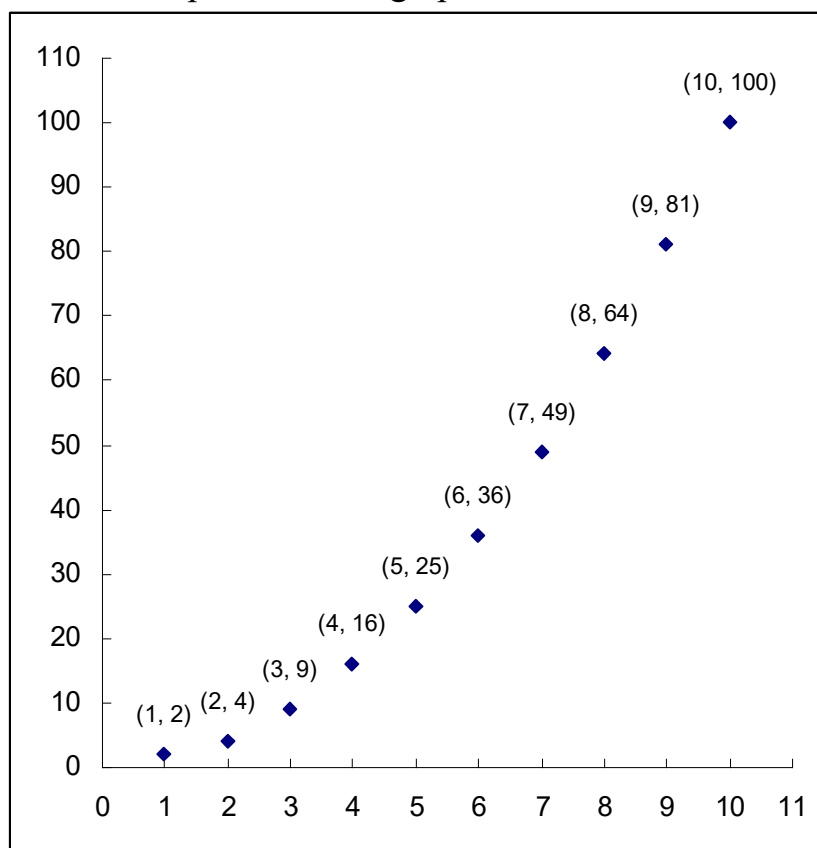
All the points on one side of the line will make the inequality true; all the points on the other side of the line will make the inequality false.

Function Notation

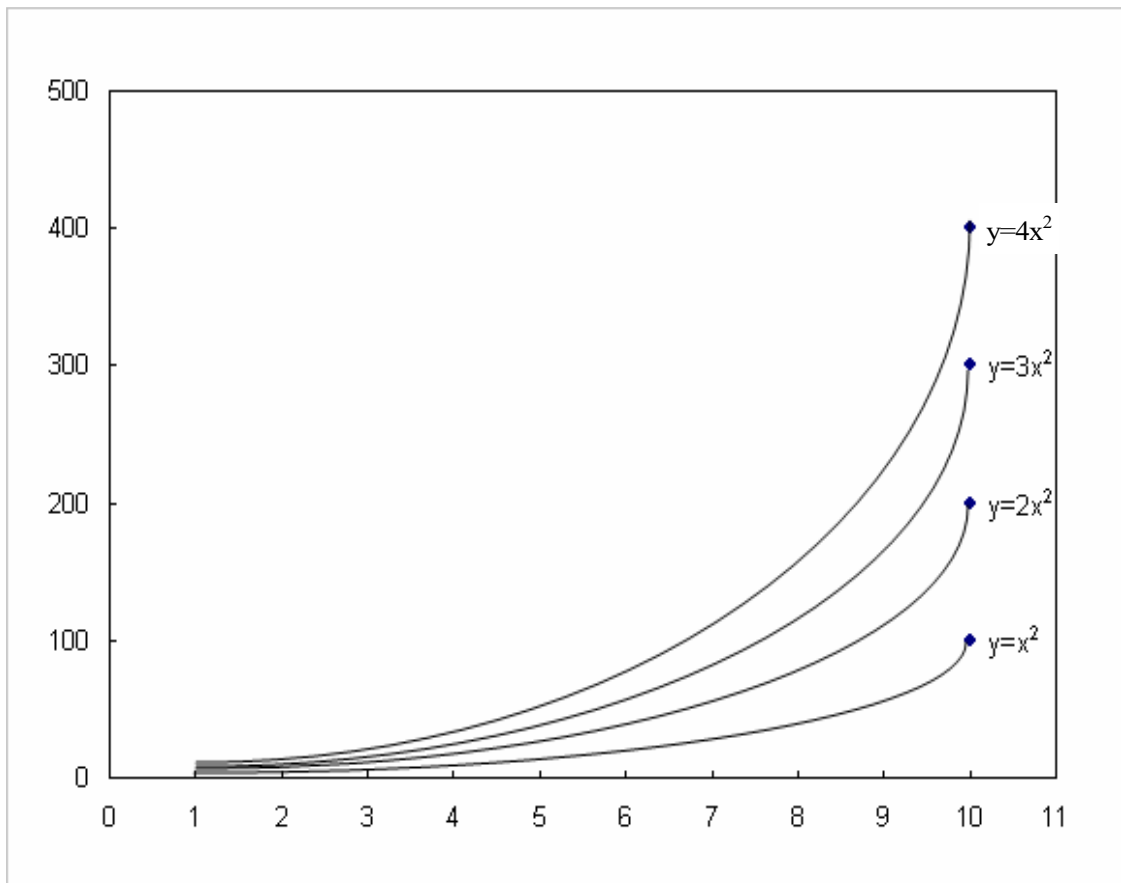
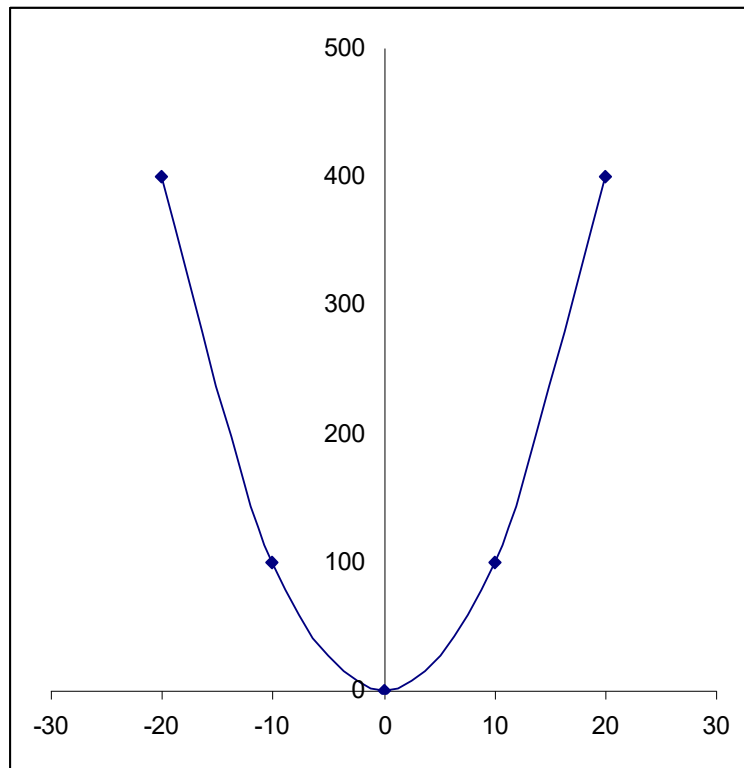
We worked on some more examples of graphs of functions. If x represents the width of a square, then $A = f(x) = x^2$ represents the area. We created a table of values for the function.

X	A = f(x)
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100

Next, we plotted these points on the graph.



Function Notation



Function Notation

We drew one graph showing the functions $y = x^2$, $y = 2x^2$, $y = 3x^2$, and $y = 4x^2$, so we could see how they are related.

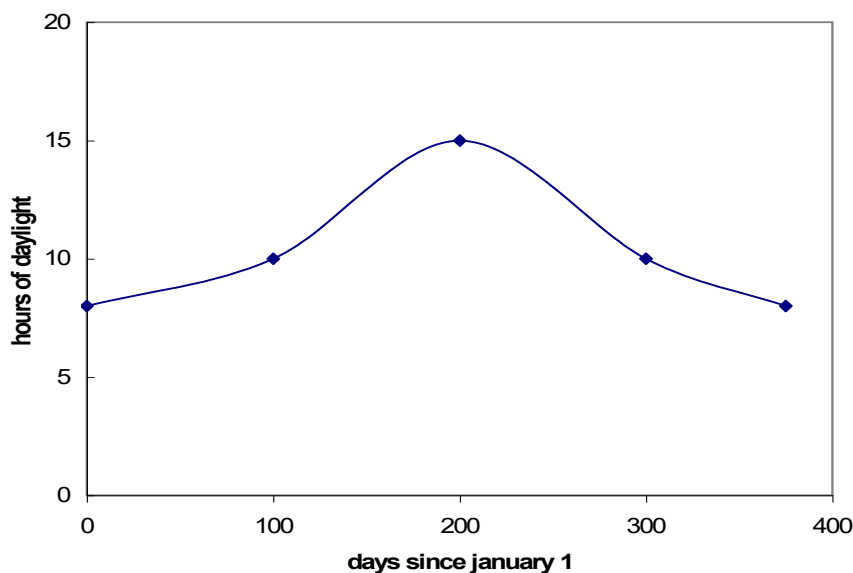
These functions belong to the same family. It was decided to call $y = x^2$ the parent function and recognized that there would be many functions of similar form but with different values multiplying the x^2 .

Next, we graphed the function $y = f(x) = x^3$.

More About Functions

A function $f(x)$ can be specified in one of these four ways:

- Starting a formula (this is usually the best way to represent the function, if it is possible to find a formula).
- Listing the value of the function for every possible value in the domain. This is not practical if the domain is large (as is the case if the domain contains all real numbers), but sometimes listing the value of the function for a set of values that you are interested in will help clarify the nature of the function.
- Graphing the function. Typically the independent variable (called x unless there is a more appropriate name for a particular situation) is measured along the horizontal axis, and the dependent variable (typically called y) is measured along the vertical axis. For a particular value x_1 , the height of the curve (or line) at that point gives the value of $y = f(x_1)$. Obviously the graph cannot show all possible real number values of x , but if the graph shows all the values of x that you are interested in, you can read off the value of y directly from the graph (although reading values from the graph is usually less accurate than calculating values from the formula, if a formula is available).



Function Notation

Activity 1

Let $f(x) = x^2$. Then evaluate.

- | | | | |
|-----------|-----------|---------------|---------------|
| 1. $f(0)$ | 3. $f(2)$ | 5. $f(q)$ | 7. $f(x + 2)$ |
| 2. $f(1)$ | 4. $f(a)$ | 6. $f(a + b)$ | 8. $f(-1)$ |

Let $f(x) = x^2 + 2x + 2$. Then evaluate.

- | | |
|-------------|-----------------------|
| 9. $f(0)$ | 13. $f(a)$ |
| 10. $f(1)$ | 14. $f(x + h)$ |
| 11. $f(-1)$ | 15. $f(x + h) - f(x)$ |
| 12. $f(3)$ | |

Let $f(x) = \sqrt{x}$. Then evaluate.

- | | |
|------------|------------------|
| 16. $f(4)$ | 18. $f(3a^2b^2)$ |
| 17. $f(7)$ | 19. $f(ab^2c^3)$ |

Let $f(x) = \sqrt{1+x}$. Then evaluate.

- | | |
|-------------|----------------|
| 20. $f(15)$ | 24. $f(0.5)$ |
| 21. $f(10)$ | 25. $f(0.1)$ |
| 22. $f(5)$ | 26. $f(0.01)$ |
| 23. $f(1)$ | 27. $f(0.001)$ |

Let $f(x) = 1 + \frac{1}{2}x$. Then evaluate.

- | | |
|-------------|----------------|
| 28. $f(15)$ | 32. $f(0.5)$ |
| 29. $f(10)$ | 33. $f(0.1)$ |
| 30. $f(5)$ | 34. $f(0.01)$ |
| 31. $f(1)$ | 35. $f(0.001)$ |

36. Do you notice any relationship between the two functions $f(x) = \sqrt{1+x}$ and $f(x) = 1 + \frac{1}{2}x$ as the value of x becomes small?

37. Strictly speaking, it is not correct to say that the function $g(x) = \sqrt{x}$ is the inverse of the function $f(x) = x^2$ unless you add one restriction. What is that restriction?

A function is said to be an odd function if $f(-x) = -f(x)$ for all values of x . A function is said to be an even function if $f(-x) = f(x)$ for all values of x . Are the following functions odd, even, or neither?

- | | | |
|------------------|----------------------|-------------------------------|
| 38. $f(x) = 20x$ | 40. $f(x) = x^3$ | 42. $f(x) = 6x^5 + 4x^3 + 6x$ |
| 39. $f(x) = x^2$ | 41. $f(x) = x^2 + x$ | |

Theme II : Solving First-Degree Equations, Polynomials and Inequalities

Lesson 1 : Simplifying and Solving Linear Equations (First-Degree Equations in One Variable)



Do You Know?

Algebra is that part of mathematics which studies relations and statements in which letters and symbols are used to represent specific numbers or the value of specific type numbers. **Algebraic expressions** are statements or phrases which utilize letters and symbols to represent numbers. An equation is said to be **linear** if the largest exponent of the variable is 1.

Example: $x - 3 + 3(x + 5) = 4(2x - 7)$

An **equation** is a statement with an equal sign between two algebraic expressions. An **identity** is an equation that is always true.

Example: $7 + x = 7 + x$; $14/7 + 5 = 7$

A **conditional equation** is an equation that is true when the value of the variable meet the conditions of the equation.

Example: $3x + 6 = 18$; $1/3x + 15 = 25$

A **linear equation** is a conditional equation with the highest degree of the variable, usually x , being one (1). After being simplified, the **general form** or **standard form** of a linear equation is $ax + b = 0$ where a and b are real numbers and a is not equal 0.

Examples: $x + 5 = 8$ or $x - 3 = 0$
 $x + (x + 3) + 2x = 23$ or $4x - 20 = 0$

A linear equation, in standard form, also represent the equation of a line.

Simplifying and Solving Linear Equations

Rules used for algebraic expressions and linear equations

While the equations in Example 4-2 are simple enough to solve by inspection, it's not usually that easy. So we need a set of rules or operations that we can use to solve first-degree equations.

- If $A(x)$, $B(x)$ and $C(x)$ are algebraic expressions, then the algebraic equation $A(x) = B(x)$ may be expressed as

Rule 1. $A(x) + C(x) = B(x) + C(x)$ When equals are added to equals the results are equal.

Rule 2. $A(x) - C(x) = B(x) - C(x)$ When equals are subtracted from equals the results are equal.

Rule 3. $A(x) \cdot C(x) = B(x) \cdot C(x)$ When equals are multiplied by equal, the result are equal if the numbers or expressions being multiplied is not zero.
where $C(x) \neq 0$

Rule 4. $\frac{A(x)}{C(x)} = \frac{B(x)}{C(x)}$ When equals divided by equal, the results are equal if all numbers and expressions being divided are not zero.
where $C(x) \neq 0$

Equivalent equations are two equations that have identical solutions. In solving linear or first-degree equations we use the properties of algebraic operations to obtain different equivalent equations. This process is continued until a simple equivalent equation is obtained, which should lead to a solution.

Example: Solve $6x + 3 = 3x - 7$

Solution Produce equivalent equations, using the rules, to obtain a solution:

$$6x + 3 = 3x - 7$$

$$(6x + 3) - 3 = (3x - 7) - 3 \quad [\text{Rule 2: Subtract equals from equals}]$$

$$6x + 3 - 3 = 3x - 10 \quad [\text{Cancel and combine}]$$

$$6x - 3x = (3x - 10) - 3x \quad [\text{Rule 2: Subtract equals from equals}]$$

$$3x = 3x - 10 - 3x \quad [\text{Cancel and combine}]$$

$$\frac{3x}{3} = \frac{-10}{3} \quad [\text{Rule 4: Divide equals by equal, nonzero expressions}]$$

$$x = \frac{-10}{3}$$

Check $6\left(\frac{-10}{3}\right) + 3 = 3\left(\frac{-10}{3}\right) - 7$

$$\frac{-60}{3} + \frac{9}{3} = \frac{-30}{3} - \frac{21}{3}$$

$$\frac{-51}{3} = \frac{-51}{3}$$

Simplifying and Solving Linear Equations

Guidelines for solving linear or first-degree equations

The previous example utilizes the guidelines for solving linear equations. These guidelines can be summarized in the five following steps:

- Step 1: Add or subtract the same constant terms on both sides to bring all the constant numbers to one side. Then simplify or combine the constant.
- Step 2: Add or subtract the same variable terms on both sides to bring all the variable terms to the other side of the equation. Simplify or combine the variables.
- Step 3: Multiply or divide both sides by the same number (not equal to zero) that will lead to the variable being alone with a coefficient of 1.
- Step 4: Further simplify, if necessary to obtain the solution.
- Step 5: Substitute the solution into the original equation to check its validity.

Consider the algebraic expression

$$x - 3 + 3(x + 5) = 4(2x - 7)$$

Simplify the expression; solve the expression for x ; and check the solution for correctness.

Solution First simplify the given equation as much as possible:

$$\begin{aligned}x - 3 + 3(x + 5) &= 4(2x - 7) \\x - 3 + 3x + 15 &= 8x - 28 && \text{[Multiply out]} \\4x + 12 &= 8x - 28 && \text{[Combine]}\end{aligned}$$

- (1) Bring all the constant numbers to one side:

$$4x + 12 - 12 = 8x - 28 - 12 \quad \text{[Rule 2: Subtract 12 from each side]}$$

- (2) Bring all the variable terms to the other side:

$$\begin{aligned}4x - 8x &= 8x - 40 - 8x && \text{[Rule 2: Subtract 8x from each side]} \\-4x &= -40 && \text{[Combine]}\end{aligned}$$

- (3) Divide both sides by the same number

$$\frac{-4x}{-4} = \frac{-40}{-4} \quad \text{[Rule 4: Divide both sides by -4]}$$

to make the variable stand alone:

$$x = 10$$

Simplifying and Solving Linear Equations

(4) Check: Substitute the solution into the original equation:

$$\begin{aligned}10 - 3 + 3(10 + 5) &= 4(2 \bullet 10 - 7) \\7 + 3(15) &= 4(20 - 7) \\7 + 45 &= 4(13) \\52 &= 52\end{aligned}$$

Remark: It doesn't really matter which side you bring the variable term to—but the convention is to put variables on the left-hand side (lhs) and constants on the right-hand side (rhs).

Eliminating a fraction

If an algebraic expression involves a fraction, one can eliminate the fraction (get rid of the denominator) by multiplying each term of the expression by the number in the denominator.

Example: $\frac{x-3}{2} = 3$

$$2\left(\frac{x-3}{2}\right) = 2(3) \quad \text{[Multiply by denominator]}$$

$$2\frac{(x-3)}{2} = 2(3) \quad \text{[Simplify]}$$

$$\begin{aligned}x - 3 &= 6 \\x - 3 + 3 &= 6 + 3 & \text{[Combine]} \\x &= 9\end{aligned}$$

Check: $\frac{x-3}{2} = 3$

$$\frac{(9)-3}{2} = 3$$

$$\frac{6}{2} = 3$$

$$3 = 3$$

Simplifying and Solving Linear Equations

Least Common Denominator (LCD)

If an algebraic expression contains two (2) or more fractions, one can eliminate the fractions (the denominators) by multiplying each term of the expression by the LCD. The LCD is the smallest number for which each denominator is a divisor.

Example:
$$\frac{x+2}{4} + \frac{7x-2}{6} = 3$$

Solution:
$$\frac{x+2}{4} + \frac{7x-2}{6} = 3$$

$$\begin{aligned} 12 \frac{(x+2)}{4} + 12 \left(\frac{7x-2}{6} \right) &= 3 && \text{[Multiply by LCD = 12 Rule 3]} \\ 3(x+2) + 2(7x-2) &= 36 && \text{[Simplify]} \\ 3x + 6 + 14x - 4 &= 36 && \text{[Expand]} \\ 14x + 3x + 6 - 4 &= 36 && \text{[Combine]} \\ 17x + 2 - 2 &= 36 - 2 && \text{[Rule 1]} \\ 17x &= 34 && \text{[Simplify]} \\ x &= 2 \end{aligned}$$

Check:
$$\frac{x+2}{4} + \frac{7x-2}{6} = 3$$

$$\frac{(2)+2}{4} + \frac{7(2)-2}{6} = 3$$

$$\frac{4}{4} + \frac{12}{6} = 3$$

$$1 + \frac{12}{6} = 3$$

$$\frac{6}{6} + \frac{12}{6} = 3$$

$$\frac{6+12}{6} = 3$$

$$\frac{18}{6} = 3$$

$$3 = 3$$

Remark: The process used to solve for the unknown x can also be used to solve equations involving one or more letters besides the one whose value is being found.

Simplifying and Solving Linear Equations

Example: Solve for x in $mx - 3a = 7a$

Solution:

$$\begin{aligned} mx - 3a &= 7a \\ mx - 3a + 3a &= 7a + 3a && \text{[Rule 1]} \\ mx &= 7a + 3a && \text{[Simplify]} \\ \frac{mx}{m} &= \frac{7a + 3a}{m} && \text{[Rule 4 (assuming } m \neq 0)] \\ x &= \frac{7a + 3a}{m} \end{aligned}$$

Check: $\frac{m(7a + 3a)}{m} - 3a = 7a$

$$\begin{aligned} (7a + 3a) - 3a &= 7a \\ 7a + 3a - 3a &= 7a \\ 7a &= 7a \end{aligned}$$

Not all equations have a solution. Often the only way to know is by attempting to solve the equation.

Example: Solve the first-degree equation $3(2x + 4) - 7 = 2(3x - 1)$

Solution: Multiply out the given quantities and follow the rules:

$$\begin{aligned} 3(2x + 4) - 7 &= 2(3x - 1) \\ 6x + 12 - 7 &= 6x - 2 \\ 6x + 5 &= 6x - 2 \end{aligned}$$

Subtract 6x from each side and obtain $5 = -2$. Impossible! Hence there is NO value of x that satisfies the given equation.

Simplifying and Solving Linear Equations

Activity 1

1. $3x - 12 = 0$

2. $\frac{1}{3}x + 8 = 0$

3. $2x + 9 = x - 13$

4. $7y + 8 = 3y - 16$

5. $6 + \frac{z}{5} = -\frac{2}{3}$

6. $\frac{1}{3} = \frac{u}{u+1} - \frac{4}{3}$

7. $\frac{1}{4}x + 5 = 3x - \frac{5}{2}$

8. $\frac{x-5}{2} - 7 = \frac{2-3x}{5} + \frac{3}{10}$

9. $x - (5 - x) - 3$

10. $v - (v + 4)(v - 1) = 6 - v^2$

11. $.5m - 1.3 = .7(m - .6)$

12. $\frac{1}{2r+2} - 1 = \frac{2}{r+1}$



Theme II : Solving First-Degree Equations, Polynomials and Inequalities

Lesson 2: Simplifying and Solving Inequalities and Absolute Value Equations in One Variable (First Degree)

Do You Know?

If we are concerned with the ordering of algebraic expressions, we work with inequalities. The four relationals or symbols that we use to connote inequalities for algebraic expressions are

$$\begin{array}{ll} > \text{ greater than} & \geq \text{ greater than or equal to} \\ < \text{ less than} & \leq \text{ less than or equal to} \end{array}$$

Examples of inequalities: $x + 5 > 10$, $\frac{8+y}{4} \leq 28$, and $.09 < (2r - .5) < .13$

In working with first-degree inequalities, we again will be looking for solutions – the values of the variable that make the inequality a true expression. Please note that the solution of an inequality is a set of values. This set may have a single value, a finite number of values, or a continuous string (interval) of values.

Example

Express the value(s) that will make these inequalities true:

$$(a) 4x + 3 > 15 \quad (b) 4x + 3 \geq 15$$

Solution

(a) Picking arbitrary numbers, we see that the values $x = 6$ and $x = 4$ are solutions to the inequality $4x + 3 > 15$ because

$$\left\{ \begin{array}{l} 4(6) + 3 > 15 \\ 27 > 15 \end{array} \right\} \text{ and } \left\{ \begin{array}{l} 4(4) + 3 > 15 \\ 19 > 15 \end{array} \right\}$$

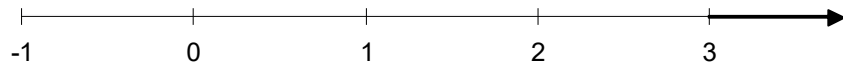
But these are not the only values. Any real number greater than 3 will work. In other words, there is a continuous string, or interval, of numbers greater than but not equal to 3 that will solve this inequality. We can graph this solution set on the number line as



Solving Inequalities and Absolute Value Equations

Where the single parenthesis “(“ at 3 indicates that 3 is not included in the set. We can also express this solution set in two other ways: We can use

- (1) set-builder notation $\{x | x > 3\}$, read as “the set of x values such that x is greater than 3” or
- (2) interval notation $(3, \infty)$, where “(“ indicates a lower bound down to but not including 3 and “ ∞)” indicates an extension of the interval without bound in the positive direction.
- (b) Everything that’s true of $4x + 3 > 15$ is also true of $4x + 3 \geq 15$, except that the set of values in the solution includes 3; ie., $4(3) + 3 = 15$. So we can graph this solution as



Rules used for solving inequalities are the same as the rules for solving equations – except that the multiplication (Rule 3) and division (Rule 4) rules must be changed:

Rule 3: If $A < B$
 then $AC < BC$ when $C > 0$
 or $AC > BC$ when $C < 0$

Multiply both sides of an inequality by the same positive number does not change direction of the inequality. Multiplying both sides by the same negative number changes the direction of the inequality.

Rule 4: If $A < B$
 then $\frac{A}{D} > \frac{B}{D}$ when $D > 0$

Dividing both sides of an inequality by the same positive number does not change direction of the inequality.

or $\frac{A}{D} < \frac{B}{D}$ when $D < 0$

Dividing both sides by the same negative number changes the direction of the inequality.

Example Solve (a) $-4x < 8$ and (b) $\frac{2-x}{3} \geq \frac{1}{4}$.

Solution

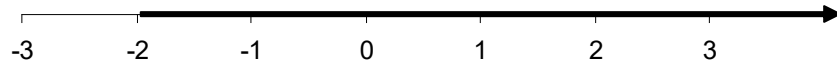
a) To solve $-4x < 8$, divide by -4 and change the direction of the inequality to obtain

$$\frac{-4x}{-4} > \frac{8}{-4}$$

Solving Inequalities and Absolute Value Equations

$$x > -2$$

The values of x that satisfy the original inequality are all those values greater than -2 , or $\{x|x > -2\}$ or $(-2, \infty)$:



- b) To solve $\frac{2-x}{3} \geq \frac{1}{4}$, first multiply by 12, the lcd, to clear the expression of fractions or the denominators and obtain

$$\frac{12(2-x)}{3} \geq \frac{12(1)}{4}$$

$$4(2-x) \geq 3$$

$$8-4x \geq 3$$

[Simplify]

$$(8-4x)-8 \geq 3-8$$

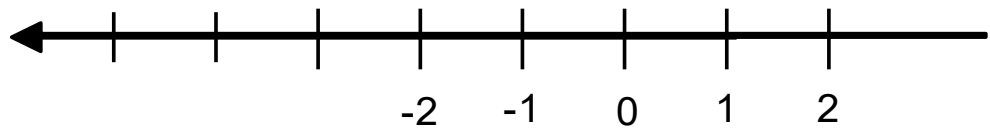
$$-4x \geq -5$$

$$x \leq \frac{5}{4}$$

[Rule 2: Subtract equals]

[Rule 4: Divide by -4 and reverse the inequality]

$$\text{So } \{x|x \leq \frac{5}{4}\} \text{ or } (-\infty, \frac{5}{4}]:$$



Solving Inequalities and Absolute Value Equations

Guidelines for solving compound inequalities

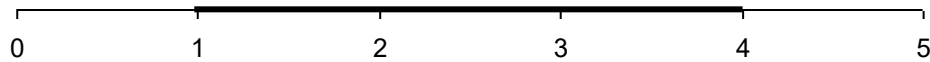
Inequalities may have more than one relational or inequality symbol. We call such algebraic expressions a compound inequality. We solve these inequalities different from regular inequalities.

Example Solve the compound inequality $2 < 6x - 4 \leq 20$.

Solution We treat this inequality as if we were working on two inequalities at the same time. Then we work to isolate x in the middle.

$$\begin{array}{rcl}
 2 < 6x - 4 \leq 20 & & \\
 2 + 4 < (6x - 4) + 4 \leq 20 + 4 & & \text{[Add to each member]} \\
 6 < 6x \leq 24 & & \text{[Simplify]} \\
 1 < x \leq 4 & & \text{[Divide by (positive) 6]}
 \end{array}$$

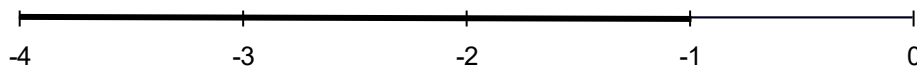
The solution is a set $\{x | 1 < x \leq 4\}$, which is an interval $(1, 4]$ of x values greater than 1 and these less than or equal to 4:



Remarks: You can use any of the various set notations available for expressing solutions of compound inequalities. The notation $\{x | 1 < x \leq 4\}$ is equivalent to the notation for the interval $(1, 4]$ and the graphical (number line pictorial is just another way of expressing the same set.

Example Solve the inequality $2 < \frac{-4x + 2}{3} < 6$.

$$\begin{array}{rcl}
 \text{Solution} & 3(2) < 3 \frac{(-4x + 2)}{3} < 3(6) & \text{[Multiply each member by +3]} \\
 & 6, \quad -4x + 2 < 18 & \\
 & 6 - 2 < -4x + 2 - 2 < 18 - 2 & \text{[Add -2 to each member]} \\
 & 4 < -4x < 16 & \\
 & \frac{4}{-4} > \frac{-4x}{-4} > \frac{16}{-4} & \text{[Divide each member by -4,} \\
 & & \text{Reversing each inequality]} \\
 & -1 > x > -4 & \\
 & -4 < x < -1 & \text{[Rewrite the inequalities]}
 \end{array}$$



Remark: In compound inequalities we usually write smaller quantities on the left and larger quantities on the right. And we never have opposing directions of inequality signs in a given inequality! (Correct: $1 < 2 < 3$; Incorrect: $3 > 2 < 4$.)

Solving Inequalities and Absolute Value Equations

Absolute-Value Equations and Inequalities

In mathematics problems we often are interested in the undirected distance between the origin and any other point x on the number line. This undirected distance is called the absolute value of x and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example, using the definition, we have $|5| = 5$, $|-5| = 5$, $|-3| = 3$, $|\frac{1}{2}| = \frac{1}{2}$, and so on.

Absolute-value equations

In general, if $|x| = d$, then $x = d$ and $x = -d$, or $x = \pm d$.

Example Solve $|x| = 3$.

Solution Since the left side of the equation is the undirected distance from the origin to a point x on the number line and this distance is 3 units, you can conclude that x has the value +3 if the point is to the right of the origin and the value -3 if the point is to the left. Thus, the solutions of $|x| = 3$ are $x = 3$ and $x = -3$.

For an algebraic expression $A(x)$, if $|A(x)| = k$, then $A(x) = k$ and $A(x) = -k$, or $A(x) = \pm k$.

Example Solve $|x - 4| = 3$.

Solution Write this equation as a pair of equations:

$$x - 4 = 3 \quad \text{and} \quad x - 4 = -3 \quad [\pm k = \pm 3]$$

$$x = 4 + 3 \quad \text{and} \quad x = 4 - 3$$

So $x = 7$ and $x = 1$

and these two values are the solutions.

$$\begin{array}{ll} \text{Check: } |7 - 4| = 3 & |1 - 4| = 3 \\ |3| = 3 & |-3| = 3 \end{array}$$

Solving Inequalities and Absolute Value Equations

Absolute-value inequalities

From the definition of absolute value, the algebraic expression

$$|ax + b| < c \quad \text{where } c > 0$$

is equivalent to the pair of inequalities

$$\begin{array}{ll} ax + b < c & \text{where } ax + b \geq 0 \\ -(ax + b) < c & \text{where } ax + b < 0 \end{array}$$

Then, if we multiply both members of $-(ax + b) < c$ by -1 , we change the direction of the inequality and obtain the equivalent expression $ax + b > -c$. Now we can draw a conclusion:

The expression $|ax + b| < c$ is equivalent to the pair

$$ax + b < c \text{ and } ax + b > -c$$

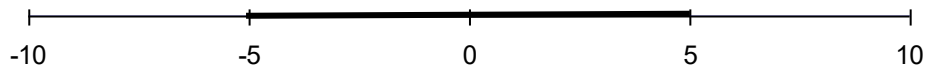
which may be written in one continuous string as $-c < ax + b < c$, or

$$|ax + b| < c \text{ is the same as } -c < ax + b < c$$

The pair of continuous string may then be algebraically changed to obtain the solution(s).

Example Solve $|x| < 5$.

Solution The solution of $|x| < 5$ are the values in the interval
 $-5 < x < 5$



The geometric interpretation of $|x| < 5$ shows the interval to be all points whose undirected distance from the origin is less than 5 units.

Example Solve $|2x - 3| < 7$.

Solution Write the inequality as the equivalent pair

$$|2x - 3| < 7 \Leftrightarrow \left\{ \begin{array}{l} 2x - 3 < 7 \\ 2x - 3 > -7 \end{array} \right\} \quad [|ax + b| < c \Leftrightarrow \left\{ \begin{array}{l} ax + b < c \\ ax + b > -c \end{array} \right\}]$$

Solving Inequalities and Absolute Value Equations

Which can be written as the continuous string

$$-7 < 2x - 3 < 7 \quad [-c < ax + b < c]$$

Now follow the guidelines and we get

$$-7 + 3 < 2x - 3 + 3 < 7 + 3 \quad [\text{Add 3 to each member}]$$

$$-4 < 2x < 10 \quad [\text{Simplify}]$$

$$-\frac{4}{2} < \frac{2x}{2} < \frac{10}{2} \quad [\text{Divide by (positive) 2}]$$

$$-2 < x < 5$$

The solution values are all x in the open interval graphed as



Any x value in this solution interval may be placed into the original algebraic expression to see that the original inequality is satisfied.

We solve the equivalent pair separately when the inequality is $>$ or \geq .

Example Solve and graph on the number line $\left| \frac{x-3}{4} \right| \geq 2$.

Solution This inequality is equivalent to the pair of inequalities

$$\frac{x-3}{4} \geq 2 \quad \text{if} \quad \frac{x-3}{4} > 0 \quad (\text{is nonnegative})$$

And $-\left(\frac{x-3}{4}\right) \geq 2 \quad \text{if} \quad \frac{x-3}{4} < 0 \quad (\text{is negative})$

Solving Inequalities and Absolute Value Equations

Now we solve each one separately to get two solutions:

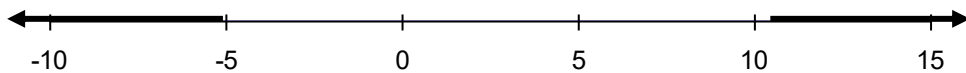
$$\frac{x-3}{4} \geq 2 \qquad -\left(\frac{x-3}{4}\right) \geq 2$$

$$x-3 \geq 8 \qquad \frac{x-3}{4} \leq 2$$

$$x \geq 11 \qquad x-3 \leq -8$$

$$x \leq -5$$

So the solution set is $\{x \mid x \leq -5 \text{ and } x \geq 11\}$:



Solving Inequalities and Absolute Value Equations

Activity 1

1. Solve the inequality $(4x - 2) - (5 - 3x) \leq 3 + 2x$ and graph the solution.
2. Solve the inequality $5(x + 4) - 3(x + 7) \geq 5$ and graph the solution.
3. Solve the inequality $-2(x - 1) + 4(x + 6) \leq 0$ and graph the solution.
4. Solve the inequality $2x - 8 = -x + 3(x - 5)$ and graph the solution.
5. Solve the inequality $4(x - 1) + 6 < 2(x - 1) + (3 - 5x)$ and graph the solution.
6. Solve the compound inequality $-5 < 3z + 4 < 16$ and graph the solution.
7. Solve the compound inequality $3 + x < 5x - 2 < x + 13$ and graph the solution.
8. Solve the absolute value equation $|3x - 4| = 8$ and graph the solution.
9. Solve the absolute value equation $|-2x + 3| + 5 = 11$ and graph the solution.
10. Solve the absolute value inequality $|4x + 3| < 9$ and graph the solution.
11. Solve the absolute value inequality $|x - 4| > 17$ and graph the solution.
12. Solve the absolute value inequality $|x - 3| + 5 < 8$ and graph the solution.

Theme II : Solving First-Degree Equations, Polynomials and Inequalities

Lesson 3 : Simplifying and Solving Quadratic Equations and Inequalities in One Variable (Second-Degree)



Do You Know?

An equation that is in—or can be placed into—the form $ax^2 + bx + c = 0$, where a , b and c are constants and $a \neq 0$, is a **second-degree** or **quadratic equation** in x . Simplifying and solving a quadratic equation involve finding the value(s) of x that will satisfy the equation. These values are known as **roots** of the equation, and the set of roots is the solution. A second-degree equation always has exactly two roots:

- Both roots may be real and different having two real values in the solution set.
- Both roots may be real and the having only one real value in the solution set.
- Both roots may be complex numbers having no real values in the solution set.

There are three standard ways of solving second-degree equations: factoring, completing the square, and using the quadratic formula.

Solving quadratic equations by factoring

The factoring method depends on the **zero product principle**, namely:

If $A \cdot B = 0$, then either $A = 0$ or $B = 0$ (or both A and B are zero).

If a quadratic expression can be factored, we can apply this principle by setting each factor equal to zero and finding the value of x from the resulting linear equations.

Solving Polynomials, Quadratic Equations and Inequalities

Example Solve the quadratic equation $x^2 + 5x + 4 = 0$ by the factor method.

Solution Factor the left-hand side:

$$x^2 + 5x + 4 = 0$$

$$(x + 4)(x + 1) = 0$$

so either $x + 4$ or $x + 1$ must equal zero. Thus

$$\text{if } x + 4 = 0, \text{ then } x = -4$$

$$\text{if } x + 1 = 0, \text{ then } x = -1$$

Check	$(-4)^2 + 5(-4) + 4 = 0$	$(-1)^2 + 5(-1) + 4 = 0$
	$16 - 20 + 4 = 0$	$1 - 5 + 4 = 0$
	$0 = 0$	$0 = 0$

There are two real values of x that satisfy the equation; so $x = -4$ and $x = -1$ are the roots of the equation and the solution is the set $\{x | x = -4, x = -1\}$.

Example Solve $x^2 - 5x = 0$

Solution The left-hand side of this equation is a quadratic expression (where $a = 1$, $b = -5$, and $c = 0$) and can be factored:

$$x^2 - 5x = 0 \qquad x(x - 5) = 0$$

So by the zero product rule,

$$x = 0 \qquad \text{or} \qquad x - 5 = 0$$
$$\qquad \qquad \qquad \qquad \qquad \qquad x = 5$$

and the solution set is $\{x | x = 0, x = 5\}$.

Check	$0^2 - 5 \bullet 0 = 0$	$5^2 - 5 \bullet 5 = 0$
	$0 - 0 = 0$	$25 - 25 = 0$
	$0 = 0$	$0 = 0$

Solving Polynomials, Quadratic Equations and Inequalities

Example Solve $x^2 - 16 = 0$

Solution
$$\begin{aligned}x^2 - 16 &= 0 \\(x - 4)(x + 4) &= 0\end{aligned}$$

Thus
$$\begin{array}{ccc}x - 4 = 0 & \text{or} & x + 4 = 0 \\x = 4 & & x = -4\end{array}$$

Remark We could also have written this as

$$\begin{aligned}x^2 &= 16 \\ \sqrt{x^2} &= \sqrt{16} \\ x &= \pm 4 \text{ which is the same result as before.}\end{aligned}$$

When a quadratic equation is in the form $(Ax + B)^2 = C$, where A, B, and C are constants and $A \neq 0$, we can solve without factoring first. To do this, we use the fact that

$$\text{if } H^2 = K^2, \text{ then } H = \pm K$$

And, just as we can multiply and divide equals, we can take the square roots of equals.

Example Solve $(x - 4)^2 = 9$

Solution Take the square root of both sides:

$$\begin{aligned}(x - 4)^2 &= 9 \\ \sqrt{(x - 4)^2} &= \sqrt{9} \\ x - 4 &= \pm 3 \\ x &= 4 \pm 3 \\ x = 4 + 3 & \quad \text{or} \quad x = 4 - 3 \\ = 7 & \quad \quad \quad = 1\end{aligned}$$

Check
$$\begin{array}{ccc}(7 - 4)^2 = 9 & & (1 - 4)^2 = 9 \\ 3^2 = 9 & & (-3)^2 = 9\end{array}$$

The roots of equations in the form $(Ax + B)^2 = C$ may not be real: The solution set may include complex numbers. Recall that $\sqrt{-1} = i$ and that $x^2 = -1$.

Solving Polynomials, Quadratic Equations and Inequalities

Example Solve $(2x + 5)^2 = -9$

Solution $(2x + 5)^2 = -9$
 $2x + 5 = \pm \sqrt{-9}$ [Take the square root of both sides]
 $= \pm \sqrt{-1} \sqrt{9}$
 $= \pm 3i$

Thus $2x = -5 \pm 3i$
 $x = -\frac{5}{2} \pm \frac{3}{2}i$ [Solutions]

Check

Let $x = -\frac{5}{2} + \frac{3}{2}i$. Then

$$\begin{aligned}(2x + 5) &= 2\left(-\frac{5}{2} + \frac{3}{2}i\right) + 5 \\ &= -5 + 3i + 5 \\ &= 3i\end{aligned}$$

Let $x = -\frac{5}{2} - \frac{3}{2}i$. Then

$$\begin{aligned}(2x + 5) &= 2\left(-\frac{5}{2} - \frac{3}{2}i\right) + 5 \\ &= -5 - 3i + 5 \\ &= -3i\end{aligned}$$

Substitute into the original equation:

$$\begin{array}{ll}(3i)^2 = -9 & (-3i)^2 = -9 \\ 9i^2 = -9 & 9i^2 = -9 \\ -9 = -9 & -9 = -9\end{array}$$

Solving quadratic equations by completing the square

We can learn from factoring another method for solving quadratics. Many quadratic equations can be changed from the form $ax^2 + bx + c = 0$ to form $(Ax + B)^2 = C$, so that the equation then may be solved by taking square roots. This process—called **completing the square**—involves adding the proper term to both sides of the given equation in order to complete the perfect trinomial square on the left. There are seven steps needed to solve an equation by completing the square:

- Step 1: Divide all the terms of the equation by the coefficient of x^2 .
- Step 2: Subtract the constant term (c/a) from both sides.
- Step 3: Calculate the number equal to one-half the coefficient of x ; then square it.
- Step 4: Add the value calculated in Step 3 to both sides.
- Step 5: Rewrite the left-hand side trinomial as a binomial squared.
- Step 6: Simplify the right-hand side.
- Step 7: Solve by extracting the roots.

Solving Polynomials, Quadratic Equations and Inequalities

Example Solve $2x^2 + 8x - 3 = 0$ by completing the square.

Solution Follow the seven (7) steps outlined:

$$2x^2 + 8x - 3 = 0$$

$$x^2 + 4x - \frac{3}{2} = 0 \quad [\text{Step 1: Divide each term by the coefficient of } x^2]$$

$$x^2 + 4x = \frac{3}{2} \quad [\text{Step 2: Subtract the constant term from each side}]$$

Find $\frac{1}{2}(4) = 2$ and calculate $2^2 = 4$. [Step 3: Find one-half the coefficient of x and square it]

$$x^2 + 4x + 4 = \frac{3}{2} + 4 \quad [\text{Step 4: Add 4 to each side}]$$

$$(x + 2)^2 = \frac{3}{2} + 4 \quad [\text{Step 5: Rewrite } x^2 + 4x + 4 \text{ as a binomial squared}]$$

$$(x + 2)^2 = \frac{11}{2} \quad [\text{Step 6: Simplify the rhs}]$$

$$x + 2 = \pm \sqrt{\frac{11}{2}} \quad [\text{Rule 7: Extract the roots}]$$

$$x = -2 \pm \sqrt{\frac{11}{2}}$$

Solving Polynomials, Quadratic Equations and Inequalities

Example Solve $x^2 - 7x - 5 = 0$ by completing the square.

Solution $x^2 - 7x - 5 = 0$

$$x^2 - 7x = 5 \quad \text{[Step 1 and 2]}$$

$$x^2 - 7x + \left(\frac{7}{2}\right)^2 = 5 + \left(\frac{7}{2}\right)^2 \quad \text{[Step 3 and 4]}$$

$$x^2 - 7x + \left(\frac{7}{2}\right)^2 = 5 + \frac{49}{4} \quad \text{[Simplify]}$$

$$\left(x - \frac{7}{2}\right)^2 = \frac{69}{4} \quad \text{[Steps 5 and 6]}$$

$$x - \frac{7}{2} = \pm \sqrt{\frac{69}{4}} \quad \text{[Step 7]}$$

$$x = \frac{7}{2} \pm \frac{\sqrt{69}}{2} \quad \text{[Simplify]}$$

$$x = \frac{7 \pm \sqrt{69}}{2}$$

Solving Polynomials, Quadratic Equations and Inequalities

Example Solve the general quadratic equation $ax^2 + bx + c = 0$ by completing the square.

Solution

$$ax^2 + bx + c = 0$$

$$x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$$

[Divide by the coefficient of x^2]

$$x^2 + \left(\frac{b}{a}\right)x = -\frac{c}{a}$$

[Subtract the constant term]

$$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

[Compute the value to be added and simplify]

$$x^2 + \left(\frac{b}{a}\right)x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

[Add the above value to both sides]

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$$

[Rewrite each side (lcd = $4a^2$)]

$$= \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

[Take square roots of both sides]

$$= \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$= \frac{\pm \sqrt{b^2 - 4ac}}{2a}$$

[Simplify]

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

[Solve for x]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

[Solutions to $ax^2 + bx + c = 0$]

Solving Polynomials, Quadratic Equations and Inequalities

Solving quadratic equations by using the quadratic formula

The following example provides the result we need to solve the general quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$. The solutions are written entirely in terms of the coefficients a , b , c :

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ it is called the quadratic formula.}$$

When we can not use any other method to solve a given quadratic equation, we can always use the quadratic formula.

Remark: Memorize these formulas. They will be used many times in future math studies.

Example: Use the quadratic formula to solve $x^2 - 6x + 9 = 0$

Solution: Here we see that $a = 1$, $b = -6$, and $c = 9$. Then we find the value of $b^2 - 4ac$ to help our computations:

$$\begin{aligned} b^2 - 4ac &= (-6)^2 - 4(1)(9) \\ &= 36 - 36 \\ &= 0 \end{aligned}$$

So the values of x are

$$x = \frac{-(-6) + \sqrt{0}}{2} = 3 \text{ and } x = \frac{-(-6) - \sqrt{0}}{2} = 3$$

Here the roots are equal, so there is just one value in the solution set, $\{x \mid x = 3\}$.

Example: Use the quadratic formula to solve for x in $3x^2 - 5x - 7 = 0$.

Solution: Here $a = 3$, $b = -5$, and $c = -7$; so

$$\begin{aligned} b^2 - 4ac &= 25 - 4(3)(-7) \\ &= 25 + 84 \\ &= 109 \end{aligned}$$

The solutions are

$$x = \frac{5 + \sqrt{109}}{6} \text{ and } x = \frac{5 - \sqrt{109}}{6}$$

Solving Polynomials, Quadratic Equations and Inequalities

Quadratic-Type Equations

Frequently, we encounter equations that can be rewritten in the form of a quadratic equation, so they can be solved by one of the familiar quadratic methods.

Equations that have the variable under a radical (square root) sign can be rewritten by squaring both sides.

Example Solve $\sqrt{x+3} = x+1$

Solution Begin by squaring both sides:

$$\sqrt{x+3} = x+1$$

$$(\sqrt{x+3})^2 = (x+1)^2$$

$$x+3 = x^2 + 2x + 1$$

Then subtract $(x+3)$ from both sides and simplify:

$$0 = x^2 + 2x + 1 - x - 3$$

$$0 = x^2 + x - 2$$

$$0 = (x+2)(x-1)$$
$$x = -2 \text{ or } x = 1$$

Thus

Since we are multiplying by terms involving the unknown x , we must be sure to check each root: We do this by substituting each root into the original equation.

Check

If $x = -2$:

$$\sqrt{(-2)+3} = -2+1$$

$$\sqrt{1} = -1$$

$$1 \neq -1$$

If $x = 1$:

$$\sqrt{1+3} = 1+1$$

$$\sqrt{4} = 2$$

Solving Polynomials, Quadratic Equations and Inequalities

Quadratic Inequalities

When we solve a quadratic inequality,

$$ax^2 + bx + c > 0$$

the roots of the quadratic equation $ax^2 + bx + c = 0$ play a crucial role.

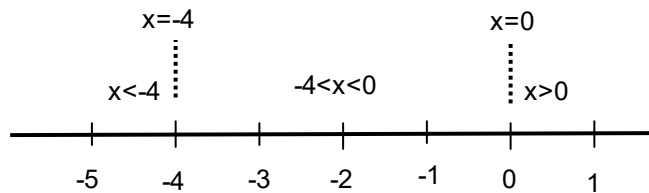
To obtain the solution set to a quadratic inequality, we determine the sign of the quadratic polynomial in each of the intervals into which the number line is divided by the roots (or zeros) of the polynomial. Then we pick the correct interval(s) as the solution.

Example Solve $x^2 + 4x \geq 0$

Solution

$$\begin{aligned} x^2 + 4x &\geq 0 \\ x(x + 4) &\geq 0 \end{aligned}$$

The zeros are at 0 and -4, so the intervals are



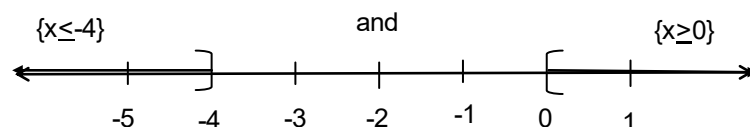
Test values:

$$x = -5 \quad x = -2 \quad x = 1$$

The polynomial at the test points has the value

$$\begin{aligned} (-5)(-5 + 4) &= (-5)(-1) = 5 && \text{at } x = -5 \\ (-2)(-2 + 4) &= (-2)(2) = -4 && \text{at } x = -2 \\ (1)(1 + 4) &= 5 && \text{at } x = 1 \end{aligned}$$

The polynomial is positive on the two outside intervals and is equal at the points $x = 0$ and $x = -4$, so the solution intervals are



Solving Polynomials, Quadratic Equations and Inequalities

Activity 1

1. Solve $x^2 - 11x + 24 = 0$

2. Solve $2x^2 - x - 3 = 0$

3. Solve $x^2 - 7x + 6 = 0$

4. Solve $3x^2 - x = 4$

5. Solve $x^2 - 2x + 5 = 0$

6. Solve $3x^2 - 9x - 12 = 0$

7. Solve $(2x + 3)^2 = 10$

8. Solve $-x^2 + 7x - 2 = 0$

9. Solve $x^2 + 2x - 8 \leq 0$

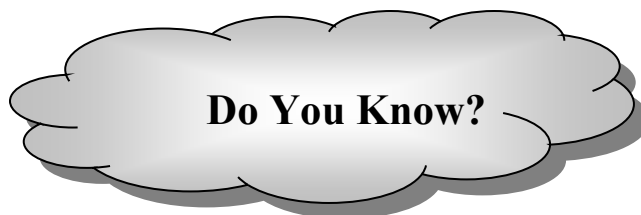
10. Solve $2x^2 > 3 - 5x$

11. Solve $(2x - 1)^2 = 16$

12. Solve $x^2 - 25 = 0$

Theme II : Solving Equations and Inequalities (Linear Equations, Quadratic Equations and Polynomials)

Lesson 4 : Working with Polynomials



Polynomials Terminology

In algebraic expressions, some of the members can have arbitrary (or unknown) values while other members have fixed (or known) values:

A letter (such as x) used to denote an arbitrary member of a set is called a **variable**.

A letter or number (such as K or 5 or $\frac{16}{3}$ or π) that has a fixed, or known, value is called a **constant**.

If n is a natural number and x is any real number, then we can write an expression x^n for a **positive power** of x (where x is the base and n is the exponent), such that

$$x^n = \underbrace{x \bullet x \bullet x \bullet \dots \bullet x}_{n \text{ times}}$$

For example, $4^3 = 4 \bullet 4 \bullet 4$ and $(-2)^4 = (-2)(-2)(-2)(-2)$, while $5^1 = 5$. Also, $x^0 = 1$ for any nonzero value of x ; that is, the value of any real number raised to the power of 0 is always 1.

The basic building block of algebraic expressions is the **monomial** cx^n , where

c is a constant (c is any real number)

x is a variable

n is a nonnegative integer ($n = 0, 1, 2, 3, \dots$)

We refer to the constant in the monomial as the **coefficient**. The expressions $3x^4$ and $\frac{16}{3}y^5$ are monomials, whose coefficients are 3 and $\frac{16}{3}$ and whose variables are x and y , respectively. But $\frac{3}{x}$ and $3\sqrt{x}$ are not monomials, because the variable in a monomial must not have a negative or rational (fractional) exponent;

Working with Polynomials

Remark: A negative exponent on any number, such as x^{-1} , indicates the reciprocal of the number; i.e., $x^{-1} = \frac{1}{x}$.

A rational exponent, such as $x^{\frac{1}{2}}$, indicates a root, so $x^{\frac{1}{2}} = \sqrt{x}$.

The sum of a finite number (one or more) of monomials is called a **polynomial**. The monomials in a polynomial are called terms of the polynomial. So a monomial cx^n is a polynomial with one term. And if we add two monomials ax^m and bx^n , we get a **binomial** $ax^m + bx^n$, which is a polynomial with two terms.

Remark: It's understood here that a, b, and c are constants (coefficients) and that the exponents m and n are nonnegative integers.

Example Explain why the following are polynomials:

(a) 2^2y

(c) $5z^2 + 4^{-2}z + 3$

(b) $5x - 3$

(d) $\sqrt{5}x^4 - \frac{3}{2}x^2 + 4x$

Solution Each of these expressions is made up of one or more monomials with the form cx^n :

- (a) The expression 2^2y is a monomial with coefficient $c = 2^2 = 4$ and exponent 1 on the variable y.
- (b) The expression $5x - 3$ is the difference of two monomials, i.e., the sum of $5x$ and -3 , and hence is a polynomial with two terms, or a binomial.
- (c) Each term is a monomial since the exponent on the variable x is a nonnegative integer and the coefficients 5, 4^{-2} , and 3 are real.
- (d) The coefficients $\sqrt{5}$, $-\frac{\pi}{2}$, and 4 are real and the exponents on the variable x added to form a polynomial.

Remark: There is no restriction that a term in a polynomial must have just one variable. An algebraic expression such as $3x^2y^2$ is a monomial in two variables (x and y) and $-6xy^2z^4$ is a monomial in three variables (x, y, and z). These multivariable monomials may also be used to build polynomials.

Working with Polynomials

Example

Explain why the following are NOT polynomials:

(a) $\frac{2}{x} - 5$

(b) $6xy^{-2} + 3x^2y$

(c) $\frac{x}{y} + \frac{5z}{7} + 5^{-2}$

(d) $s^2 + 2t^{1/2}$

(e) $s^2 + 2\sqrt{t}$

Solution

(a) No variable is allowed in the denominator: $\frac{2}{x} - 5 = 2x^{-1} - 5$.

(b) Negative exponents are not allowed on the variable.

(c) No variable is allowed in the denominator.

(d) No fractional exponents are allowed on a variable.

(e) A variable under a radical sign is not allowed: $s^2 + 2\sqrt{t} = s^2 + 2t^{1/2}$.

Classifying polynomials

We begin classifying polynomials by the number of variables they have. A polynomial may have many variables, or it may have just one. After we determine how many variables there are, we can classify polynomials by **degree**:

- The **degree of a polynomial in one variable** is the highest exponent that is found on the (single) variable with a non-zero coefficient.
The polynomial $x^2 + 6x + 9$ has one variable, x , and a degree of 2, because the highest exponent on the single variable x is 2.
- The **degree of a polynomial in more than one variable** is the highest sum of exponent on the variables in any one term.
Thus, the polynomial $x^4y^2 + 4x^3y^2 + 2xy$ has two variables, x and y , and a degree of 6, because the exponents of the variables x and y in the first term add up to 6—a sum that exceeds the sum of the exponents of the variables in the second term (5) and the third term (2).

The general form of a polynomial in one variable x is

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where each coefficient a_i is a real number. If $a_n \neq 0$, then this polynomial has degree n .

Working with Polynomials

Operations on Polynomials

A. Addition and subtraction

Since polynomials are made up of terms representing real numbers, all of the rules for the operations on real numbers will apply to them. Thus, if we have two polynomials, $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $Q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$, we can find their sum $S(x)$ and difference $T(x)$ by grouping their like terms and adding or subtracting the coefficients of the like terms:

Example Find the sum of the polynomials $P(x) = 3x^2 + 4x - 7$ and $Q(x) = 7x^2 - 9x + 4$

Solution Set up the addition by aligning the like terms:

$$\begin{array}{r} 3x^2 + 4x - 7 \\ 7x^2 - 9x + 4 \\ \hline \end{array}$$

Add the coefficients: $10x^2 - 5x - 3$

Or, simply rearrange the terms—i.e., group like terms—and use the properties for real numbers:

$$\begin{aligned} P(x) + Q(x) &= (3x^2 + 4x - 7) + (7x^2 - 9x + 4) \\ &= (3x^2 + 7x^2) + (4x - 9x) + (-7 + 4) && \text{[Group like terms]} \\ &= (3 + 7)x^2 + (4 - 9)x - 7 + 4 && \text{[Add coefficients]} \\ &= 10x^2 - 5x - 3 \end{aligned}$$

B. Multiplication

In order to multiply polynomials, we need to extend the definition of an exponent so we can multiply numbers with the same base raised to a power. To do this, we use the rule of exponents $x^n \bullet x^m = x^{n+m}$; that is,

$$x^n \bullet x^m = \underbrace{x \bullet x \bullet x \cdots x}_{n \text{ times}} \bullet \underbrace{x \bullet x \bullet x \cdots x}_{m \text{ times}} = x^{n+m}$$

note: This rule can be extended to $x^n \div x^m = x^{n-m}$ and $(x^m)^n = x^{m \bullet n} = x^{mn}$. Thus, $x^6 \div x^3 = x^{6-3} = x^3$ and $(x^2)^3 = x^{2 \bullet 3} = x^6$.

Now we can find the product of two polynomials $P(x) \bullet Q(x)$ using the distributive laws and rules of exponents.

Working with Polynomials

Example Find the product of $P(x) = x^2 + 3$ and $Q(x) = x + 4$

Solution Set up the multiplication the old “arithmetic” way:

$$\begin{array}{r}
 P(x) = \quad x^2 + 3 \\
 \times Q(x) = \quad \times x + 4 \\
 \hline
 \quad \quad + 4x^2 \quad + 12 \\
 \quad \quad \quad \quad x^3 \quad + 3x
 \end{array}$$

[Multiply each member of the top expression by 4]

[Multiply each member of the top expression by x]

$$P(x) \cdot Q(x) = \overline{x^3 + 4x^2 + 3x + 12}$$

[Add]

Or, use the distributive law directly:

$$\begin{aligned}
 P(x) \cdot Q(x) &= (x^2 + 3) \cdot (x + 4) \\
 &= x(x^2 + 3) + 4(x^2 + 3) \\
 &= (x^3 + 3x) + (4x^2 + 12) \\
 &= x^3 + 4x^2 + 3x + 12
 \end{aligned}$$

[Distributive law]

The product of three or more polynomials

The product of three polynomials is obtained in a two-step process: We find the product of two of the polynomials first; then we multiply this “intermediated” product by the third polynomial to get the final product.

$$\begin{aligned}
 P(x) \cdot Q(x) \cdot R(x) &= [P(x) \cdot Q(x)] \cdot R(x) \\
 &= P(x) \cdot [Q(x) \cdot R(x)]
 \end{aligned}$$

Product of two binomials

There is a memory device we can use to find the product of two binomials. As a model, consider the product $(2x + 3)(x + 5)$. Using the distributive laws, we obtain

$$\begin{aligned}
 (2x + 3)(x + 5) &= 2x(x + 5) + 3(x + 5) \\
 &= 2x \cdot x + 2x \cdot 5 + 3 \cdot x + 3 \cdot 5 \\
 &= 2x^2 + 10x + 3x + 15 \\
 &= 2x^2 + 13x + 15
 \end{aligned}$$

Working with Polynomials

Examining the steps in this multiplication closely, we see that

$2x^2$ comes from multiplying the First terms in each binomial, $2x$ and x
 $-10x$ comes from multiplying the Outside terms, $2x$ and -5
 $+3x$ comes from multiplying the Inside terms, 3 and x
 -15 comes from multiplying the Last terms in each binomial, 3 and -5

The method of multiplying First, Outside, Inside, and Last terms of two binomials is the standard way to multiply two binomials.

Factoring

Factoring is the process of writing a given polynomial as the product of two or more lower-degree polynomials; i.e., factoring is the reverse of multiplication.

1. Special products

There are certain special products that occur often in algebraic manipulation. Some of these special products are multiplied out, or expanded.

Special factors of special products	
Factors	Product
$(x - y)(x + y)$	$= x^2 - y^2$
$(x + y)^2$	$= x^2 + 2xy + y^2$
$(x - y)^2$	$= x^2 - 2xy + y^2$
$(x + a)(x + b)$	$= x^2 + (a + b)x + ab$
$(x - a)(x - b)$	$= x^2 - (a + b)x + ab$
$(x + y)^3$	$= x^3 + 3x^2y + 3xy^2 + y^3$
$(x - y)^3$	$= x^3 - 3x^2y + 3xy^2 - y^3$
$(x + a)(x^2 - ax + a^2)$	$= x^3 + a^3$
$(x - a)(x^2 - ax + a^2)$	$= x^3 - a^3$

Although there are other factorization formulas that could be written, most are special cases of those listed above.

Working with Polynomials

2. Factorization formulas

If a polynomial is written as the product of two or more polynomials of lower degree, then each product polynomial is called a **factor** of the original polynomial. The process of factoring is an important operation in algebra because we often need simpler expression to analyze.

3. Factoring guidelines

If every term of the polynomial contains a **common factor**, pull that term out first and then attempt to reduce the remaining part.

Example Factor (a) $x^3 - 4x^2 + 3x$ and (b) $3x^3 + 5x^2$

Solution

(a) The common factor is x , so

$$x^3 - 4x^2 + 3x = x(x^2 - 4x + 3)$$

But the trial-and-error technique provides further reduction since $x^2 - 4x + 3 = (x - 3)(x - 1)$. Thus

$$x^3 - 4x^2 + 3x = x(x - 3)(x - 1)$$

(b) Here every term contains x^2 , so we factor that part out:

$$3x^3 + 5x^2 = x^2(3x + 5)$$

If a common factor appears in each of a group of terms in the polynomial, pull this common factor out and rearrange the remaining part. This technique is called **factoring by grouping**.

Example Factor $(2x + 1)x^2 - 4(2x + 1)$

Solution

The common factor in each group is $2x + 1$. Thus

$$(2x + 1)x^2 - 4(2x + 1) = (2x + 1)(x^2 - 4)$$

Working with Polynomials

Example Factor (a) $4a^2 - 25$, (b) $64u^4 - 9v^2$, and (c) $(m+n)^2 - 16$

Solution

$$(a) 4a^2 - 25 = (2a)^2 - (5)^2 = (2a - 5)(2a + 5)$$

$$(b) 64u^4 + 9v^2 = (8u^2)^2 - (3v)^2 = (8u^2 - 3v)(8u^2 + 3v)$$

$$(c) = (m+n)^2 - 16 = (m+n)^2 - 4^2 = [(m+n) - 4][(m+n) + 4]$$

Factoring a perfect square:

$$x^2 + 2xy + y^2 = (x + y)(x + y)$$
$$x^2 - 2xy + y^2 = (x - y)(x - y)$$

Example Factor (a) $x^2 + 6x + 9$, (b) $x^2 + 4xy + 4y^2$, and (c) $9x^2 - 30xy + 25y^2$

Solution

(a) Identify 9 as the square of 3, notice that the multiplier of the middle term is twice 3, and apply the factorization formula:

$$x^2 + 6x + 9 = (x + 3)(x + 3)$$

(b) Similarly, identify $4y^2$ as the square of $2y$ and write

$$x^2 + 4xy + 4y^2 = (x)^2 + 2x(2y) + (2y)^2 = (x + 2y)(x + 2y)$$

(c) Again, identify the perfect squares first:

$$9x^2 - 30xy + 25y^2 = (3x)^2 - 30xy + (5y)^2$$
$$= (3x)^2 - 2(3x)(5y) + (5y)^2 = (3x - 5y)(3x - 5y)$$

Working with Polynomials

Activity 1

Problem Determine the type and degree of each of the following polynomials:

- (a) $x^2 - 5x + 3$ (e) $2xy^2 + 3$
(b) $y - 2y^2 + \sqrt{6}y + 7$ (f) $x^3 + 5x^2y + 8$
(c) $3z^4 - 9z^2 + 4$ (g) $12x^2y^3z + 27xy^5z^4 + 19x^4y^2z^2$
(d) 5^2

Problem Find the sum and difference of each polynomial pair:

- (a) $P(x) = x^2 - 4x + 2$, $Q(x) = 3x^2 + 5x - 1$
(b) $S(y) = \frac{1}{2}y^2 + 8y - 3$, $T(y) = \frac{1}{8}y^2 + \frac{3}{4}y - 7$
(c) $F(x) = 7x^4 - 9x^3 + 2x - 8$, $Q(x) = 3x^4 + x^3 - 8x^2 + 1$
(d) $S(t) = 16t^2 + \frac{3}{4}t - \frac{9}{8}$, $V(t) = \frac{5}{8}y + 5$

Problem Find the product of each polynomial pair:

- (a) $F(x) = x^2 + 4$, $G(x) = x - 6$
(b) $P(x) = x^2 + 3x - 5$, $Q(x) = x^2 + x + 1$
(c) $M = x^2 + xy$, $N = xy + y^2$

Problem Find the product of these binomials:

- (a) $(2x + 1)(x - 3)$ (c) $(x + 3y)(2x - y)$
(b) $(x + 4)(5x - 1)$ (d) $(7x - 4y)(2x + 5y)$

Problem Find each special product:

- (a) $(x + 2y)^3$ (d) $(r^2 + 3s)^3$
(b) $(x + 3)(2x + 4)(3x + 5)$ (e) $\left(\frac{1}{u} - \frac{1}{v}\right)^2$
(c) $(2a - 3b)^3$

Working with Polynomials

Activity 1 [cont.'d]

Problem Factor the following polynomials and find their roots:

(a) $4x^2 - 9$

(c) $x^2 + 8x + 16$

(b) $x^2 - 7x + 12$

(d) $(2a + b)^2 - 25$

Problem Factor the following polynomials and find their roots:

(a) $y^2 + 2y - 15$

(c) $r^5 - 9r^3$

(b) $t^4 + t^3 - 20t^2$

(d) $x^2 + 2xy - 8y^2$

Problem Factor the following polynomials:

(a) $x^3 - 3x^2 - 2x + 6$

(c) $(x^2 - 4) + 3(x + 2)$

(b) $x^2(x + 3) - 4x - 12$

(d) $9x^2 + 6x + 1$

Problem Factor the following polynomials:

(a) $x^3 - 27$

(c) $x^2 - 4x + 4 - 9y^2$

(b) $8y^3 + 1$

(d) $t^9 + 1$

Problem Solve these equations for the unknown

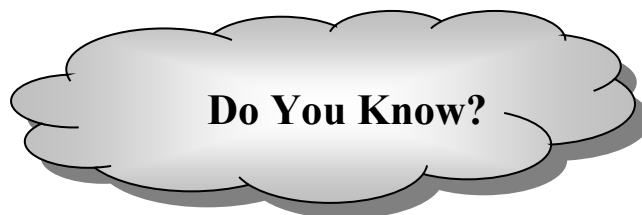
1. $x^3 - x = 0$ (Hint: Factor an x out of each term.)
2. $x^3 - 2x = 0$
3. $2x^3 - 3x^2 - 5x = 0$
4. $3x^5 + 4x^4 - 10x^3 = 0$
5. $y^2 - 13y + 36 = 0$
6. $x^4 - 13x^2 + 36 = 0$
7. $x^4 - 2x^2 - 35 = 0$
8. $x^6 + 4x^3 - 96 = 0$
9. $x^{10} - 33x^5 + 32 = 0$
10. $x^3 - 9x^2 + 20x = 0$
11. $x^3 - 2x^2 - 35x = 0$

Problem In each exercise, make up a polynomial equation that has the solutions indicated

12. $x = 1, x = 2, \text{ or } x = 3$
13. $x = -3, x = 4, \text{ or } x = 6$

Theme II : Solving Equations and Inequalities (Linear Equations, Quadratic Equations and Polynomials)

Lesson 5: Applications – Solving Real Problems with Algebra



How to Solve Real Problems with Algebra

Algebra is more than symbol manipulation. You also have to apply the theory and techniques you learn to practical situations. In fact, applications (word or real) problems are the most important part of algebra. There are some general steps that can help you to solve word (real) problems:

- Step 1: Pick a letter to represent some quantity that you're looking for in the problem.
- Step 2: Create other expressions, using this same letter, to represent other quantities in the problem.
- Step 3: Develop an equation or inequality to connect the quantity you want with quantities you know.

Note: Any equation (or inequality) will do: You do not have to find an equation that says $x = [\text{something}]$ – that's what algebraic manipulations are for!

- Step 4: Solve this equation using the algebraic techniques that are easiest for you.
- Step 5: Check your answer(s) to see if your results satisfy all the conditions of the problem; i.e., make sure you've answered the right question(s).

Steps 1 through 3 are the hardest. These require the translation of English words into the language of mathematics. For example:

"The sum of x and y " $\Leftrightarrow x + y$

"Three times the sum of p and q " $\Leftrightarrow 3(p + q)$

"The larger minus the smaller" $\Leftrightarrow L - s$

"The sum of twice m and 6" $\Leftrightarrow 2m + 6$

"The sum of a and b is more than 10" $\Leftrightarrow a + b > 10$

"The sum of a number and 12 is twice the number" $\Leftrightarrow n + 12 = 2n$

Applications – Solving Real Problems with Algebra

A Real Problem Involving First-Order Equations

Example: A salesperson earned \$150 more commission in the second month of employment than in the first month. In the third month the commission was double that of the second month. The three-month total commission was \$1650. How much commission did the salesperson earn in the first month? the second month? the third month?

Solution

Step 1: Let x be the first month's commission.

Step 2: Then $x + 150$ was the commission in the second month and $2(x + 150)$ was the commission in the third month.

Step 3: $x + (x + 150) + (2x + 200) = 1650$

Step 4: $x + 450 = 1650$
 $4x = 1200$
 $x = 300$

Step 5: The commissions were

Month 1	$x = \$ 300$
Month 2	$x + 150 = \$ 450$
Month 3	$2(x + 150) = \$ 900$
Total	\$1650

A Real Problem Involving First-Order Inequalities

Example: To be eligible for a contest, you have to be over a certain age. Twice your age plus 3 years must be more than your age plus 18 years. How old must you be?

Solution

Step 1: Let A denote your age.

Step 2: The twice your age is $2A$ and twice your age plus 3 years is $2A + 3$.

Step 3: The conditions say that "twice your age plus 3 years must be more than your age plus 18 years," so $2A + 3 > A + 18$

Step 4: $(2A + 3) - A > (A + 18) - A$
 $A + 3 > 18$
 $(A + 3) - 3 > (18) - 3$
 $A > 15$

Step 5: So you need to be older than 15 years to participate.

Remark: Setting up Steps 1 – 5 helps in the solution process.

Applications – Solving Real Problems with Algebra

Example: The scores on your first three tests in algebra were 88, 92, and 83. If it takes an average of 90 or above on the four semester tests to get an A for the course, how high must your score be on the last test to ensure an A?

Solution

Step 1: Let x be the score you need to get.

Step 2: Then you must have an average of $\frac{(88+92+83+x)}{4}$

Step 3: $\frac{(88+92+83+x)}{4} \geq 90$

Step 4: $88 + 92 + 83 + x \geq 360$
 $x + 263 \geq 360$
 $x \geq 97$

Step 5: So a grade of $x \geq 97$ will get you an A.

Real Problems Involving Quadratic Equations

Example: The sum of a number and five times its reciprocal is 6. What is the number?

Solution

Step 1: Let the number be x .

Step 2: Its reciprocal is $1/x$.

Step 3: You are told that the sum of x and five times $1/x$ is six, so

$$x + 5\left(\frac{1}{x}\right) = 6$$

Step 4: $(x)(x) + (x)\left(\frac{5}{x}\right) = (x)(6)$

$$\begin{aligned}x^2 + 5 &= 6x \\x^2 - 6x + 5 &= 0 \\(x - 5)(x - 1) &= 0\end{aligned}$$

Step 5: Hence the number is $x = 5$ or $x = 1$: There are two solutions.

$$\begin{array}{l} \text{Check: } 5 + 5\left(\frac{1}{5}\right) = 6 \qquad 1 + 5\left(\frac{1}{1}\right) = 6 \\ \qquad 5 + 1 = 6 \qquad \qquad 1 + 5 = 6 \end{array}$$

Remark: When the problem produces a quadratic equation, we may use any of our available methods to find the solutions.

Applications – Solving Real Problems with Algebra

Example: The length of a rectangle is 8 inches greater than its width and the area is 128 square inches. Find the length and width.

Solution

Step 1: Let x be the width in inches.

Step 2: Then $x + 8$ expresses the length.

Step 3: Since the area is length times width, you have

$$x(x + 8) = 128$$

$$x^2 + 8x = 128$$

or

Step 4: $x^2 + 8x - 128 = 0$

Using the quadratic formula:

$$x = \frac{-8 \pm \sqrt{64 - 4(1)(-128)}}{2 \cdot 1}$$

$$= \frac{-8 \pm \sqrt{64 + 512}}{2}$$

$$= \frac{-8 \pm \sqrt{576}}{2}$$

$$= \frac{-8 \pm 24}{2}$$

Thus $x = \frac{-8 + 24}{2}$ or $\frac{-8 - 24}{2}$

$$= \frac{16}{2} \quad \frac{-32}{2}$$

$$= 8 \quad = -16$$

Step 5: The second values, $x = -16$, is rejected because the width cannot be negative. So

$$\text{Width} = x = 8$$

$$\text{Length} = x + 8 = 16$$

Check: area = (width)(length)

$$= (8)(16)$$

$$= 128$$

Applications – Solving Real Problems with Algebra

Real Problem Involving Quadratic Inequalities

Problem The sum of the squares of two consecutive positive integers is greater than or equal to 41. Find the number pairs for which this condition is true.

Solution

Step 1: Let one of the consecutive positive integers be x .

Step 2: Let the other integer be $x + 1$.

Step 3: The sum of their squares is

$$x^2 + (x + 1)^2 \geq 41$$

Step 4:

$$x^2 + (x^2 + 2x + 1) \geq 41$$

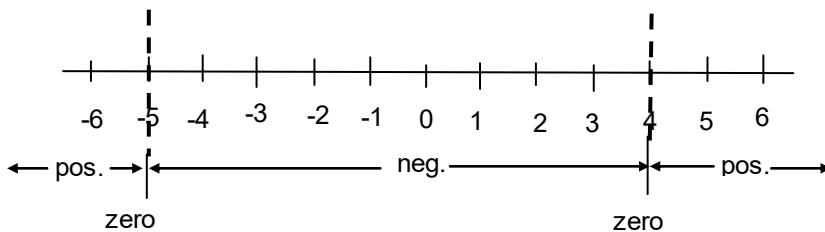
$$2x^2 + 2x + 1 \geq 41$$

$$2x^2 + 2x \geq 40$$

$$2x^2 + 2x - 40 \geq 0$$

$$x^2 + x - 20 \geq 0$$

$$(x + 5)(x - 4) \geq 0$$



Step 5: The solution set to the inequality is $x \leq -5$ or $x \geq 4$. But the numbers must be positive, so you discard $x \leq -5$ and retain the set $x \geq 4$. Hence the number pairs satisfying the condition are

Applications – Solving Real Problems with Algebra

Example Solving a Physics Problem

If an object is propelled upward with initial velocity of 96 feet/second from an initial height of 240 feet, ignoring air resistance, the height h of the object in feet as a function of time t in seconds is given by $h(t) = -16t^2 + 96t + 240$. Factor the greatest common factor from this function and use the factored form to find the height of an object 5 seconds after it is propelled upward.

Factoring, we obtain

$$\begin{aligned}h(t) &= -16t^2 + 96t + 240 \\ &= -16(t^2 - 6t - 15).\end{aligned}$$

Substitute for t or obtain the height after 5 seconds.

$$\begin{aligned}H(5) &= -16(5^2 - 6(5) - 15) \\ &= -16(25 - 30 - 15) \\ &= -16(-20) = 320\end{aligned}$$

Thus, the object is 320 feet above the ground after 5 seconds.

Applications – Solving Real Problems with Algebra

Activity 1

1. The sum of two consecutive odd integers is 32. What are the numbers?
2. If a car salesperson makes 150 dollars per week in salary and 40 dollars in commission on each car sold, how many cars must she sell in a week to have a gross income of more than 400 dollars a week?
3. One positive integer is 4 more than twice another positive integer, and the product of the integers is 70. Find these integers.
4. The sum of an integer and twice its reciprocal is $\frac{2}{2}$. Find the number.
5. An object is thrown straight up from the ground with an initial speed of 64 feet/second. The formula for the height above the ground at any time t is $h(t) = 64t - 16t^2$. (a) When is the object 48 feet high? (b) When does the object return to the ground?
6. A boy can ride his bicycle 4 times as fast as he can walk. On a 2-hour trip to grandmother's, he rode for 12 miles and walked 3 miles. What is his bicycle-riding speed?
7. When 5 is subtracted from 4 times a number, the result is the same as the sum of the number and 4. Find the number.
8. Nancy is four years older than Deborah. In 10 years the sum of their ages will be 64. How old are they now?
9. The sum of two consecutive integers is 13 less than 5 times the larger. What are the integers?
10. The price of a rug is directly proportional to its area. If a circular rug 5 feet in diameter sells for 60 dollars, what would the price be for a rug 9 feet in diameter? [Hint: $A = \pi r^2$.]
11. The sum of two numbers is 3 and the difference of their squares is 4. What are the numbers?
12. The monthly profit from a microwave-oven sales outlet is given by $P = 12S - .04S^2 - 50$ if the sales S are under 250 units.
 - (a) How much profit is there on sales of 160 units in a month?
 - (b) How many units must be sold to avoid a loss?
 - (c) How many units must be sold to earn a profit of 750 dollars?
13. If your rectangular garden is to have an area of 600 ft^2 and you have 120 ft of fence to enclose it, what should its dimensions be?
14. The sum of three numbers is 27. The sum of the first two is 14 while the sum of the last two is 21. What are they?
15. A farmer pays the same total amount for 6 cows as he does for 14 goats. If a cow costs 120 dollars more than a goat, what is the cost of (a) a cow? (b) a goat?
16. In manufacturing a certain item, company x figures that its profit P (in hundreds of dollars) is given by $P = (8 - n)^2 - 64$, where n is the number of items produced. For what values of n will the company make a profit?

Applications – Solving Real Problems with Algebra

Activity 1 [Cont.'d]

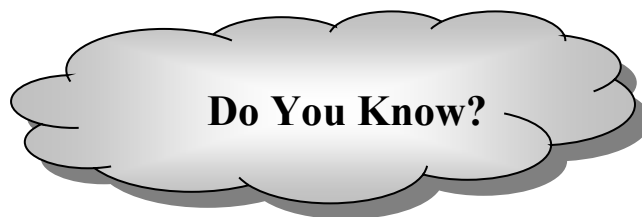
17. A retired professor has \$16,000 to invest. She wants to put part of it in a one-year certificate of deposit (C.D.) at 14% and the remainder in a money market (M.M.) fund at 12%. If her expected profit for the first year is \$2000, how much should she put into each fund?
18. Walt and Sharon rent a car for a given day at an agency that charges \$24 plus 11 cents per mile driven. They must keep the total cost for the rental under \$45. What is the maximum number of miles they may drive?
19. The denominator of a fraction is one larger than three times the numerator. If 10 is added to the numerator and 32 is added to the denominator, the value of the fraction is unchanged. What is this fraction?
20. The total income from the operation is given by the function: $h(x) = 5x^2 - 2x$ and the profit function $k(x) = x^2 + 3x - 7$.
What is the combined operation income and profit if 20 units are produced and sold?

**Grades
9 and 10**

*Division II :
Introduction to Geometry:*

Theme I : Foundation of Geometry

Lesson I : Fundamental Definitions and Important Concepts



Understanding geometry is very important for anyone who wishes to understand how to measure, how things are designed, how things are constructed, how objects relate to other objects, and how we can prove that certain things are true or not true. In fact the word geometry is derived from two words : « geo, » meaning earth and « metric, » meaning measurement. To do geometry well, one must understand and learn how to visualize, illustrate, demonstrate and properly utilize certain fundamental definitions and important concepts such as « undefined » terms, and postulates. Also fundamental definitions, equivalent definitions, and definitions of important geometric concepts such as angles, triangles, parallel lines, perpendicular lines, congruence properties, characteristics, theorems, proofs, inductive and deductive reasoning and more.

Undefined terms : These words are so fundamental and basic that they cannot be defined using simpler terms ; however, they can be described. For geometry the undefined terms are point, line, and plane.

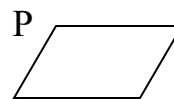
Point : a point represents a position ; it has no length, width, or depth.
Example : ● A.

Line : A line is a set of continuous points, with no spaces between them, that extends indefinitely in two opposite directions.

Example : A \longleftrightarrow B

Plane : A plane is a set of points with spaces between them, that forms a flat surface that has no depth and extends indefinitely in all directions.

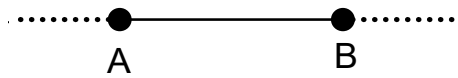
Example : A plane is usually represented by a closed four-sided figure with a single capital letter in one corner.



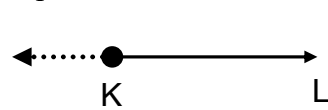
Fundamental Definitions and Important Concepts

A good definition is grammatically correct and gives all the necessary characteristics or properties to clearly identify the term or concept being defined. Some fundamental definitions for geometry will now be presented.

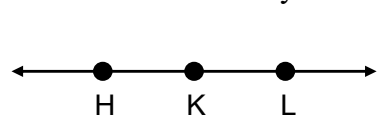
A line segment : A line segment is a part of a line, consisting of two points on the line called end points, and the set of all points between the endpoints.

Example :  denoted by \overline{AB}

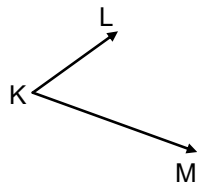
Ray : A ray is a part of a line consisting of a given point on the line, called an end point or initial point and all points on the line on one side of the end point.

Example :  denoted by \overrightarrow{KL}

Opposite Rays : Two rays in a given line, each having the same end point, each ray having different directions and the two rays form the whole line.

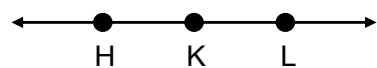
Example :  \overleftarrow{HK} and \overrightarrow{KL} are opposite rays

Angle : An angle is the set of points in a plane formed by the joining of two distinct rays with a common end point called the vertex . The rays are called the sides of the angle.

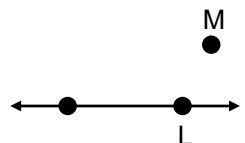


We name the angle, Angle LKM
(In naming an angle, the vertex is the middle letter)

Collinear points : Collinear points are points that lie on the same line.

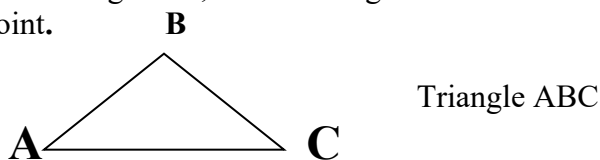
Example :  H,K,L are collinear.

Non-collinear points : non-collinear points are points that do not lie on the same line.

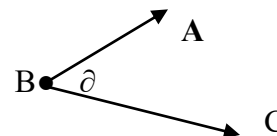
Example :  K, L, and M are non-collinear points.

Fundamental Definitions and Important Concepts

Triangle : A triangle is a figure formed in a plane by connecting three non-collinear points with three different line segments, each line segment has two of these non-collinear points as end point.



Associated Properties of an angle : Given angle ABC.

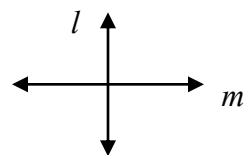
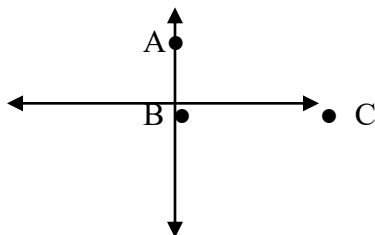


The points in the smallest area between the two rays are called interior points of the angle. The points in the plane above ray AB and below ray AC are called exterior points of the angle.

The distance between the two rays is called the interior measure of the angle. The distance between the two rays involving the exterior points is called the exterior measure of the angle. The sum of the interior measure and the exterior measure is 360 degrees.

Right Angle : If an angle is a right angle, it has the measure of 90 degrees. Four right angles are formed when two lines are perpendicular.

Example : Angle ABC is a right angle. Angle ABC has a measure of 90°

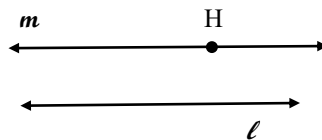


Perpendicular lines

Perpendicular Lines :

Two lines l and m are perpendicular if they form four right angles.

Parallel Lines : Two lines l and m are said to be parallel if there is a point H not on l , m passes through H (H is on m) and l and m never meet (intersect) regardless how far they are extended in either direction.



Parallel lines

Fundamental Definitions and Important Concepts

Midpoint : The midpoint of a line segment is a point that divides the segment into two equal (congruent) segments (lengths).

Betweenness : A point K is *between* points H and L on a line segment if $\overline{HL} = \overline{HK} + \overline{KL}$

Bisector (line segment) : A bisector of a line segment HK is any line segment, ray, or line that passes through the midpoint of HK.

Bisector (angle) : A bisector of an angle ABC is a ray BK that divides the angle ABC into two equal (congruent) angles ; B is the vertex of the angle and K is an interior point of the angle.

Measure (line segment) : The measure of a line segment is the « length » or distance between the two endpoints. Line segments are usually measured by a meter or by a centimeter stick.

Measure (angle) : The measure of an angle is the number of degrees between the two sides (rays) covering the interior points. Angles are usually measured by an instrument called a protractor. Measurement is stated in degrees or radians.

Congruent : Two geometric figures having the same size (measure) and same shapes are said to be congruent.

Geometry is an excellent example of a **postulational system**, a system where one begins with a set of assumptions and undefined terms that are used to develop new relationships and new knowledge that are usually expressed as theorems. Theorems are proved using undefined terms, definitions, postulates, and other previously proven theorems.

Postulate (axiom) : A postulate or an axiom is a basic assumption in geometry that is accepted as being true without any proof. Examples :

Postulate 1 : Two distinct points determine a unique line.

Postulate 2 : Three distinct non-collinear points determine a unique plane.

Inductive Reasoning : Inductive reasoning is a method of drawing conclusions by examining a few examples, observing a pattern, and then assuming that the pattern is true for all examples related to the situation.

Deductive Reasoning : Deductive reasoning uses undefined terms, definitions, and previously proven theorems to follow a step-by-step method until a valid conclusion can be reached.

Proposition : A proposition is a declarative sentence that can be proven to be true or false.

Fundamental Definitions and Important Concepts

Theorem : A theorem is a declarative statement that can be proven to be true.

Proof : A proof is a step-by-step procedure where one begins with given information and arrives at a valid conclusion after a finite number of steps; giving a reason for each step.

The if ... Then ... Sentence Structure : The If ... then ... sentence is one where an if condition is stated; after establishing the If condition it can be shown that the “then condition” (conclusion) can be shown to be true.

Example : If two lines are parallel, then they never meet.

The If ... then ... sentence is sometimes used to provide or prove an equivalent or alternative definitions for terms. Many theorems are stated in an If ... then ... sentence structure. The If ... part is said to be what is given or assumed to be true (hypothesis), the then part is said to be the conclusion or what can be proven.

Fundamental Definitions and Important Concepts

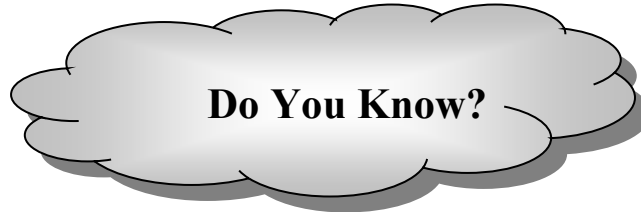
Activity I

Use the Figure above to Answer Questions for Activity I

1. Name the four rays that have K as an endpoint _____
2. Name three collinear points _____
3. Name three non-collinear points _____
4. Name four angles _____
5. Name two lines _____
6. Name two line segments _____
7. Name two triangles _____
8. Does NL appear to have a midpoint ? _____ If yes, which point ? _____
9. What two lines appear to be parallel ? _____
10. What two lines appear to be perpendicular ? _____
11. What two line segments appear to be congruent ? _____
12. What two angles appear to be congruent ? _____
13. what two angles appear to be right angles ? _____
14. Name the three sides of the triangle that includes H and K _____
15. Name the three angles of the triangle that include H and K _____

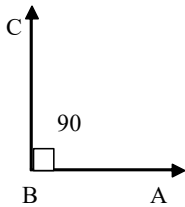
Theme I : Foundation of Geometry

Lesson 2 : Classifying Angles, Triangles, Quadrilaterals and Polygons

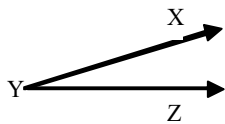


In geometry, there are some important plane figures that are known by their angles, the measurement of angles and sum of these angles. In this lesson we will discuss and study some important properties of angles, triangles, quadrilaterals, and polygons.

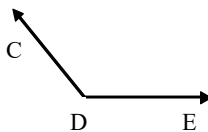
Classifying Angles



Right Angle – an angle that is exactly 90°
It is formed by two (2), perpendicular rays with a common end point.



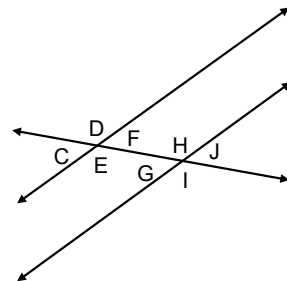
Acute Angle – an angle that is smaller than 90°



Obtuse Angle – an angle that is larger than 90° but less than a straight angle



Straight Angle – an angle whose measure is 180°



Transversal – a line that crosses 2 parallel lines
 $\angle C, \angle D, \angle I, \angle J$ (exterior angles)
 $\angle E, \angle F, \angle G, \angle H$ (interior angles)

Classifying Angles, Triangles, Quadrilaterals, and Polygons

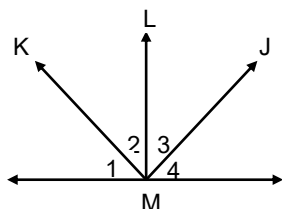


Figure A

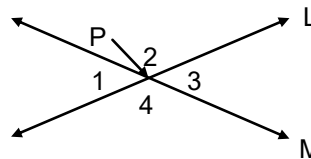


Figure B

Two angles are **complementary angles** if the sum of their measures is a right angle or 90° . In figure A, angles 1 and 2 are complementary and angles 3 and 4 are complementary.

Two angles are **supplementary angles** if the sum of their measures is a straight angle or 180° . In figure B, angles 2 and 3 are supplementary, as well as angles 4 and 3. There are other supplementary angles in figure B; what are they?

Two angles are **adjacent angles** if they have the same vertex, share a common side and have no interior points in common. In figure A, angles 2 and 3 are adjacent angles. There are other adjacent angles in figures A and B; what are they?

Two angles are **vertical angles** if they are a pair of non-adjacent (opposite) angles formed by two intersecting lines; in figure B, angles 1 and 3 are vertical angles; also angles 2 and 4 are vertical angles.

Classifying Triangles

We classify a triangle by the measurement of its angles as acute, right or obtuse. We classify a triangle according to the number of its sides that are equal (congruent) in length as being scalene, isosceles, and equilateral.

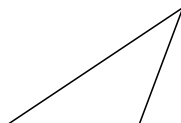
A triangle can be named by its angles.

Acute



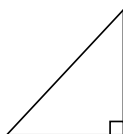
An acute triangle is a triangle with 3 acute angles.

Obtuse



An obtuse triangle is a triangle with 1 obtuse angle.

Right



A right triangle is a triangle with 1 right angle.

Classifying Angles, Triangles, Quadrilaterals, and Polygons

A triangle can be named by equal (congruent) sides.

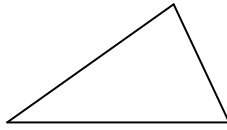


Figure A



Figure B

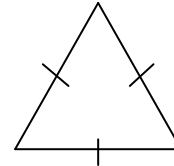


Figure C

A **scalene triangle** is a triangle that has no two or more sides of equal length and no equal angles; see figure A.

An **isosceles triangle** is a triangle that has at least two sides of equal (congruent) length and two angles of equal measure; see figure B.

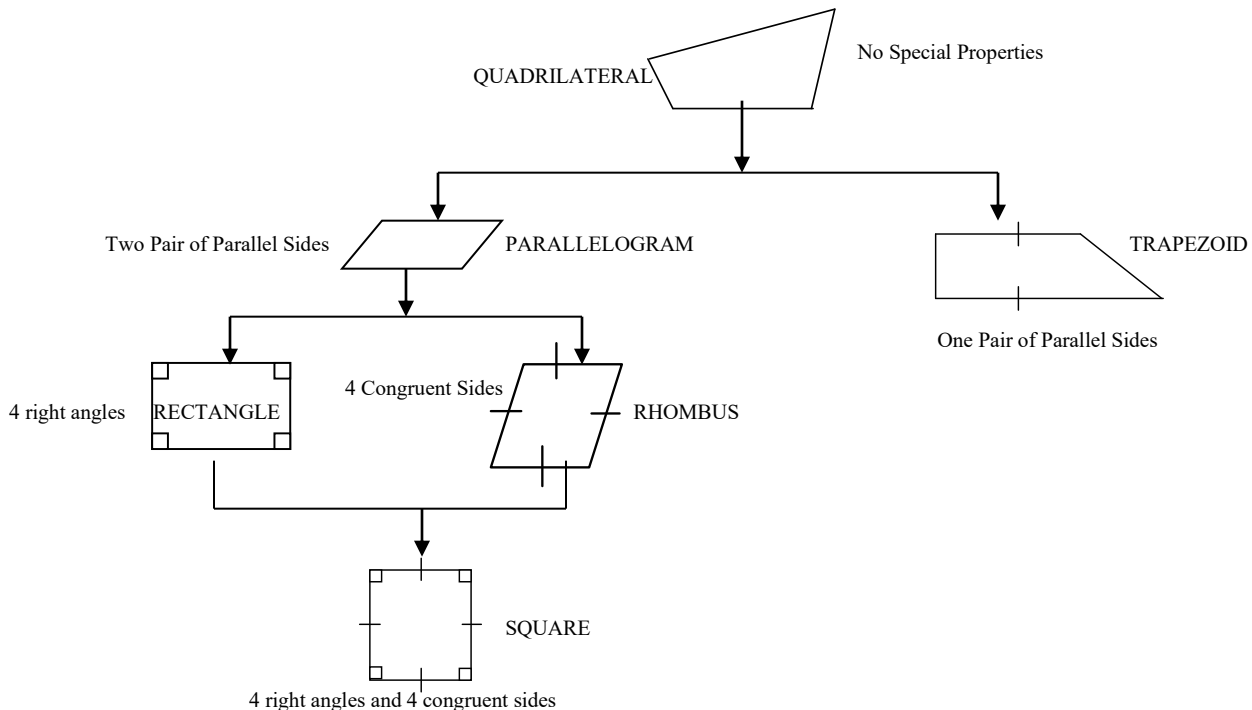
An **equilateral triangle** is a triangle in which all three sides have equal (congruent) length and all three angles have the same measure (60 degrees) ; see figure C.

Classifying Quadrilaterals

A **quadrilateral** is a closed plane figure with four (4) line segments. The relationships between the sides and the angles pairs of quadrilaterals will be studied.

The family tree of quadrilaterals shows that a quadrilateral has two major types of descendants. One, called a **trapezoid**, has exactly one pair of parallel sides. The other major type of quadrilateral has two pairs of parallel sides and is called a **parallelogram**.

The Family Tree of Quadrilaterals



Of the special quadrilaterals illustrated only the trapezoid is not a parallelogram. A rectangle, a rhombus, and a square are special types of parallelograms.

Classifying Angles, Triangles, Quadrilaterals, and Polygons

Properties of Parallelograms

Definitions of Special Parallelograms

- A **rectangle** is a parallelogram having four right angles.
- A **rhombus** is a parallelogram having four congruent sides.
- A **square** is a rectangle having four congruent sides and four right angles.

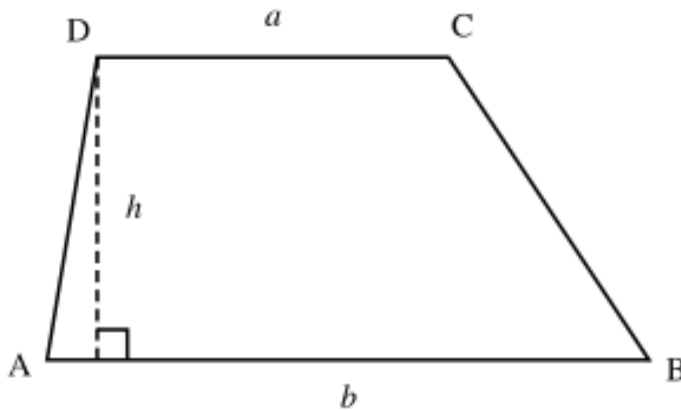
In a Parallelogram

1. Opposite sides are parallel. (Definition of parallelogram)
2. Consecutive angles are supplementary.
3. Opposite angles are congruent (equal).
4. Opposite sides are congruent (equal).
5. Diagonals bisect each other.

Properties of a Trapezoid

Definition of a Trapezoid

A **trapezoid** is a quadrilateral that has exactly one pair of parallel sides. The parallel sides are called the bases of the trapezoid. The nonparallel sides are referred to as the legs of the trapezoid.



Properties of an Isosceles Trapezoid

1. It has all the properties of a trapezoid.
2. The legs are congruent. (Definition of isosceles trapezoid)
3. The base angles are congruent.
4. The diagonals are congruent.

Classifying Angles, Triangles, Quadrilaterals, and Polygons

Summary of Properties of Diagonals of Special Quadrilaterals

Special Quadrilateral	Diagonals Are Always		Diagonals Always Bisect	
	Congruent	Perpendicular	Each Other	Vertex Angles
Parallelogram	No	No	Yes	No
Rectangle	Yes	No	Yes	No
Rhombus	No	Yes	Yes	Yes
Square	Yes	Yes	Yes	Yes
Trapezoid	No	No	No	No
Isosceles Trapezoid	Yes	No	No	No

Classifying Polygons

A polygon may be classified according to the number of sides it has. The most commonly referred to polygons are as follows:

Polygon Name	Number of Sides
Triangle	3
Quadrilateral	4
Pentagon	5
Hexagon	6
Octagon	8
Decagon	10
Duodecagon	12

A polygon may also be categorized according to whether all of its angles are equal in measure and/or all of its sides have the same length.

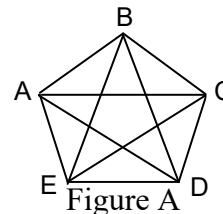
Definition of polygons having parts equal in measure

- An equiangular polygon is a polygon in which each angle has the same measure.
- An equilateral polygon is a polygon in which each side has the same length.
- A regular polygon is a polygon that is both equiangular and equilateral.

A segment that joins a pair of nonadjacent vertices of a polygon is called a diagonal. See Figure A. Clearly, there exists a relationship between the number of sides of a polygon and the number of diagonals that can be drawn.

Classifying Angles, Triangles, Quadrilaterals, and Polygons

DIAGONALS: $\overline{AC}, \overline{BD}, \overline{EC}, \overline{EB}$ and \overline{AD}



If a polygon has N sides, then $\frac{1}{2}N(N - 3)$ diagonals can be drawn.

- An n -side polygon has n vertices and n interior angles. At each vertex an exterior angle may be drawn by extending one of the sides.
- If the polygon is regular, then the measures of the interior angles are equal and the measures of the exterior angles are equal.
- The sum of the measures of the interior angles of any polygon is given by this formula:

$$Sum = 180(n - 2).$$

The sum of the measures of the exterior angles of any polygon, regardless of the number of sides, is 360.

- To find the measures of the interior and exterior angles of a regular polygon, given the number of sides (or vice versa), we use the following relationships:

$$\text{Exterior angle} = \frac{360}{n}$$

and


$$\text{Interior angle} = 180 - \text{exterior angle}$$

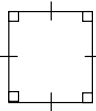
Classifying Angles, Triangles, Quadrilaterals, and Polygons

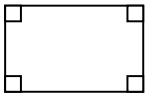
Activity 1

1. A figure (polygon) with 4 sides is a _____; the sum of its angles is _____.

2. A figure (polygon) with 3 sides is a _____; the sum of its angles is _____.

3.  _____ - opposite sides are parallel and equal

4.  _____ - a parallelogram with 4 equal sides and 4 right angles

5.  _____ - a parallelogram with 4 right angles

6.  _____ - a polygon with 6 sides

7.  _____ - a polygon with 8 sides

8.  _____ - a polygon with 5 sides

9.  _____ - a polygon with 3 sides

In Figure A,

10. Name two adjacent angles.

11. Name two vertical angles.

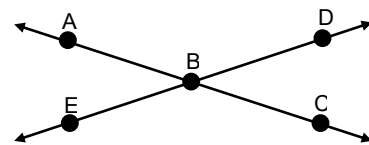


Figure A

Classifying Angles, Triangles, Quadrilaterals, and Polygons

Activity 1 [Cont.'d]

12. Name two supplementary angles.

13. Name a straight angle.

In Figure B,

14. Name a right angle.

15. Name two complementary angles.

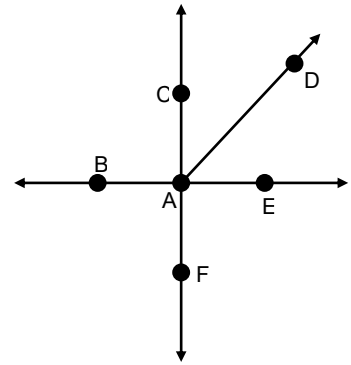


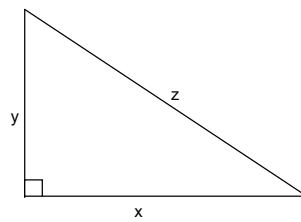
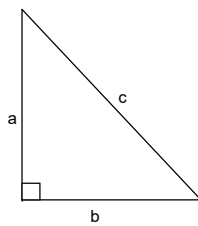
Figure B

Theme I : The Foundation of Geometry

Lesson 3 : The Right Triangle

Do You Know?

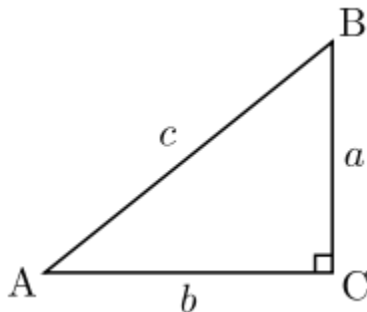
Right triangles have special properties that are very useful in solving many geometrical, mathematical and real world problems. In this lesson we will study the special properties of right triangles.



In the right triangle b or x is known as the base. In the right triangle a or y is known as the altitude. In the right triangle c or z is known as the hypotenuse.

The Pythagorean Theorem

The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs.



The sides are related by the equation

$$(\text{Hypotenuse})^2 = (\text{Leg 1})^2 + (\text{Leg 2})^2$$

This relationship, known as the **Pythagorean Theorem**, is named in honor of the Greek mathematician Pythagoras, who is believed to have presented the first proof of this theorem in about 500 B.C. Since that time, many different proofs have been offered. Here are the formal statement of the theorem:

The Right Triangle

Given a right triangle with hypotenuse c , altitude a and base b , then $c^2 = a^2 + b^2$.

Any set of three whole numbers x , y , and z is called a **Pythagorean triple** if the numbers satisfy the equation

$$z^2 = x^2 + y^2$$

The set of numbers $\{3, 4, 5\}$ is an example of a Pythagorean triple. The sets $\{5, 12, 13\}$ and $\{8, 15, 17\}$ are also Pythagorean triples. There are many others.

If $\{x, y, z\}$ is a Pythagorean triple, then so is the set that includes any whole number multiple of each member of this set. Thus, the set $\{6, 8, 10\}$ is a Pythagorean triple since each member was obtained by multiplying the corresponding member of the Pythagorean triple $\{3, 4, 5\}$ by 2:

$$\{6, 8, 10\} = \{2 \cdot 3, 2 \cdot 4, 2 \cdot 5\}$$

The following table shows additional examples.

Pythagorean Triple	Multiple of a Pythagorean Triple	Multiplying Factor
$\{3, 4, 5\}$	$\{15, 20, 25\}$	5
$\{5, 12, 13\}$	$\{10, 24, 26\}$	2
$\{8, 15, 17\}$	$\{80, 150, 170\}$	10

You should memorize the sets $\{3, 4, 5\}$, $\{5, 12, 13\}$, and $\{8, 15, 17\}$ as Pythagorean triples and be able to recognize their multiples. The need for this will become apparent as you study and use geometry more often.

Pythagorean Triples

Some commonly encountered Pythagorean triples are:

$$\begin{aligned} &\{3n, 4n, 5n\} \\ &\{5n, 12n, 13n\} \\ &\{8n, 15n, 17n\} \end{aligned}$$

where n is any positive integer ($n = 1, 2, 3, \dots$). There are many other families of Pythagorean triples, including $\{7, 24, 25\}$ and $\{9, 40, 41\}$.

The Right Triangle

Converse of the Pythagorean Theorem

If in a triangle the square of the length of a side is equal to the sum of the square of the length of the other two sides, then the triangle is a right triangle.

The 30-60 Right Triangle

Suppose the length of each side of the equilateral triangle in Figure A is represented by $2s$. If you drop an altitude from the vertex of this triangle, several interesting relationships materialize:

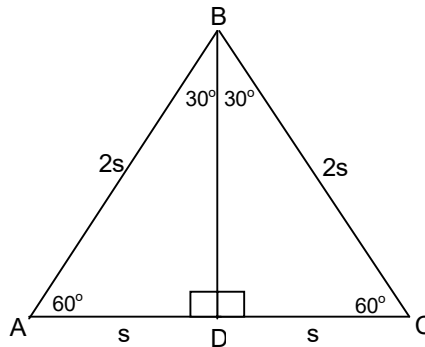


Figure A

Let's summarize the geometric information concerning $\triangle ABD$. First, $\triangle ABD$ is referred to as a 30-60 right triangle since these numbers correspond to the measures of its acute angles. In a 30-60 right triangle the following relationships hold:

30-60 Right Triangle Side Relationships

- The length of the shorter leg (the side opposite the 30 degree angle) is one-half the length of the hypotenuse:

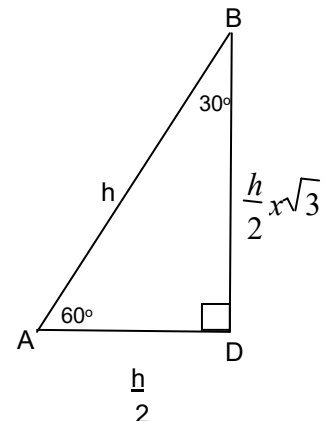
$$AD = \frac{1}{2} AB$$

- The length of the longer leg (the side opposite the 60 degree angle) is one-half the length of the hypotenuse multiplied by $\sqrt{3}$:

$$BD = \frac{1}{2} \times AB \times \sqrt{3}$$

- The length of the longer leg is equal to the length of the shorter leg multiplied by $\sqrt{3}$:

$$BD = AD \cdot \sqrt{3}$$



The Right Triangle

The 45-45 Right Triangle

Another special right triangle is the isosceles right triangle. Since the legs of an isosceles right triangle are congruent, the angles opposite must also be congruent. This implies that the measure of each acute angle of an isosceles right triangle is 45. A 45-45 right triangle is another name for an isosceles right triangle.

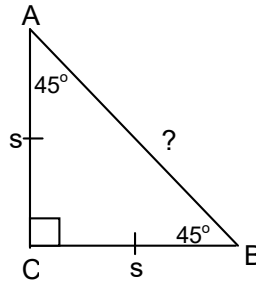


FIGURE D

To determine the relationships between the lengths of the sides of a 45-45 right triangle, we will represent the length of each leg by the letter s (Figure D).

45-45 Right Triangle Side Relationships

- The lengths of the legs are equal:

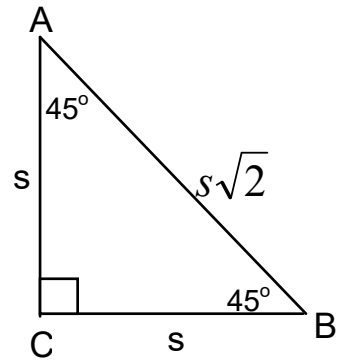
$$AC = BC$$

- The length of the hypotenuse is equal to the length of either leg multiplied by $\sqrt{2}$:

$$AB = AC \text{ (or } BC) \cdot \sqrt{2}$$

- The length of either leg is equal to one-half the length of the hypotenuse multiplied by $\sqrt{2}$:

$$AC \text{ (or } BC) = \frac{1}{2} AB \cdot \sqrt{2}$$

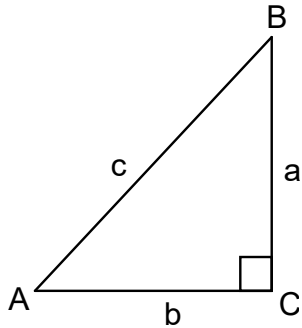


The 45-45 right triangle is called an isosceles right triangle because the two (non-right) angles are equal and the two legs are equal.

The Right Triangle

Definitions of Trigonometric Ratios

A trigonometric ratio is the ratio of the lengths of any two sides of a right triangle with respect to a given angle. Three commonly formed trigonometric ratios are called the sine, cosine, and tangent ratios:



$$\text{Sine of } \angle A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB}$$

$$\text{Cosine of } \angle A = \frac{\text{length of leg adjacent } \angle A}{\text{length of hypotenuse}} = \frac{AC}{AB}$$

$$\text{Tangent of } \angle A = \frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent}} = \frac{BC}{AC}$$

Working with Trigonometric Ratios

When working with trigonometric ratios, keep in mind the following:

- Sine, cosine, and tangent may be abbreviated as sin, cos, and tan, respectively.
- The trigonometric ratios may be taken with respect to either of the acute angles of the triangle:

$$\sin A = \frac{a}{c}$$

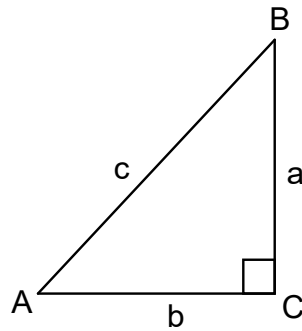
$$\sin B = \frac{b}{c}$$

$$\cos A = \frac{b}{c}$$

$$\cos B = \frac{a}{c}$$

$$\tan A = \frac{a}{b}$$

$$\tan B = \frac{b}{a}$$



- Since $\sin A$ and $\cos B$ are both equal to $\frac{a}{c}$, they are equal to each other. Similarly, $\cos A$ and $\sin B$ are equal since both are equal to $\frac{b}{c}$. Since angles A and B are complementary, the sine of an angle is equal to the cosine of the angle's complement. For example, $\sin 60^\circ = \cos 30^\circ$.

The Right Triangle

- The definitions of the three basic trigonometric ratios should be memorized.
- Three other trigonometric ratios may be taken with respect to either of the acute angles of the triangle are:

cotangent (cot), secant (sec) and cosecant (csc)

$$\cot \text{ of } \angle A = \frac{\text{length of leg adjacent } \angle A}{\text{length of leg opposite}} = \frac{AC}{BC}$$

$$\sec \text{ of } \angle A = \frac{\text{length of hypotenuse } \angle A}{\text{length of leg adjacent}} = \frac{AB}{AC}$$

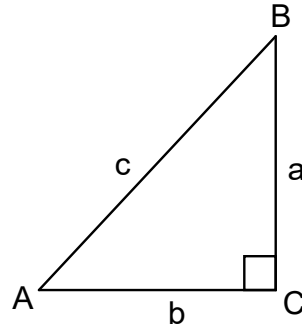
$$\csc \text{ of } \angle A = \frac{\text{length of hypotenuse } \angle A}{\text{length of leg opposite}} = \frac{AB}{BC}$$

length of leg opposite length of leg adjacent length of hypotenuse

$$\cot A = \frac{b}{a}$$

$$\sec A = c/b$$

$$\csc A = c/a$$



The Area and Perimeter of a Right Triangle

For any right triangle with hypotenuse c, base b, and altitude a:

$$\text{The Area of the right triangle } ABC = \frac{1}{2} ba$$

$$\text{The Perimeter of the right triangle } ABC = a + b + c$$

The Right Triangle

Activity 1

1. In right triangle ABC, $\angle C$ is a right angle. If $AC = 4$ and $BC = 3$, find the value of $\tan A$, $\sin A$, and $\cos A$.
2. Use the Pythagorean Theorem to determine the length of the hypotenuse for the right triangle in problem 1
3. What is the area of the right triangle in problem 1?
4. Express the six trigonometric ratios for $\angle C$
5. Calculate the perimeter of triangle ABC
6. Express the values of the $\sin 60^\circ$, $\cos 60^\circ$ and the $\tan 60^\circ$ in either radical or decimal form.
7. List six (6) different Pythagorean triples.
8. In the 30-60 right triangle ABC, if the base $b = 4$, what is
 $a = \underline{\hspace{2cm}}$ and $c = \underline{\hspace{2cm}}$?
9. In the 45-45 right triangle ABC, if the base $b = 6$, what is
 $a = \underline{\hspace{2cm}}$ and $c = \underline{\hspace{2cm}}$?
10. What is the hypotenuse of a 45 – 45 triangle whose legs are 5 units.

Niveaux
9 et 10

La Division II :
La Introduction de La Geometrie
Thème 1 :
La Foundation de La Geometrie

LEÇON D' APPOINT

Leçon 3 :
Le Right Triangle

Theme I: The Foundation of Geometry

**Lesson 4: Circles, Angle Measurement
and Associated Concepts**

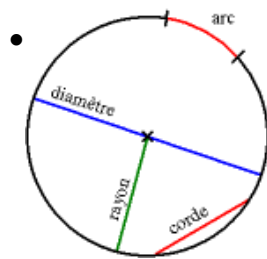


Do You Know?

Circular and curved figures frequently exist in geometry and in everyday life. In fact, circular wheels are found almost everywhere. Circles and circular figures have special properties. In this lesson many of these properties will be presented.

Some Basic Definitions:

- A circle is the set of all points in a plane with the same distance from an interior fixed point.
- The center is the interior fixed point of a circle.
- The radius is the equal distance between the center and all points on the circle.
- All radii (plural for radius) of a given circle are equal.
- A chord is a line segment whose endpoints lie on the circle.



- A diameter is a chord that passes through the center of the circle.
- An arc is a curved portion of a circle.
- A semicircle is an arc whose endpoints are on the diameter of the circle.
- A major arc is an arc that is larger than a semi-circle.
- A minor arc is an arc that is smaller than a semicircle.
- The circumference of a circle is the complete distance around a circle beginning at a point on the circle and ending at the same point. The circumference is expressed in linear units, centimeters, meters, etc.
- The interior of a circle is the set of all points whose distance from the center of the circle is less than the length of the radius of a circle.
- The exterior of a circle is the set of all points whose distance from the center of the circle is greater than the length of the radius of the circle.

Circles, Angle Measurement and Associated Concepts

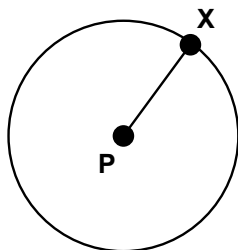


Figure A

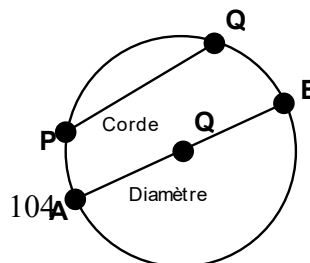


Figure B



Examples: in figures A, B, C, and D one shows examples of four different circles with several aspects of the circle being illustrated.

Examples:

- In figure A, P is the center of the circle
- In figure A, X is a point on the circle

- In figure A, the line segment PX is the radius of the circle.
- In figure A, the circumference of the circle is the distance around the circle, beginning with a point X and traveling continuously around the circle until one encounters X again.

- In figure B, the line segment AB is the diameter of the circle
- In Figure B, the distance along the circle from Q to B is a chord.

- In figure B, the distance around the circle from A to B is a semi-circle.
- In figure C, the distance around the circle from A to B is an arc.

- In figure C, from point A to C (passing through B) constitutes a minor arc.
- In figure C, from point A to point E, passing through points B, C, and D, constitutes a major arc.

- In figure C, line segment OA and OB are radii of equal length.
- In figure D, the point Y is an interior point.

Circles, Angle Measurement and Associated Concepts

- In figure D, the point Z is an exterior point
- The length of an arc of a circle is measured in linear units of measure such as centimeters, meters, etc.

- The central angle is the angle formed by the endpoints of an arc with line segments or rays between the endpoints and the center of the circle.
- The intercepted arc is the arc on the circle between the endpoints of a central angle.
- The degree of an intercepted arc is the same as the degree of the associated central angle.

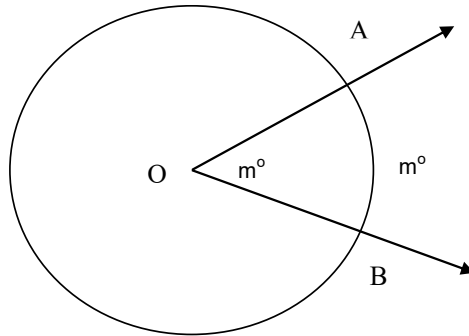


Figure A

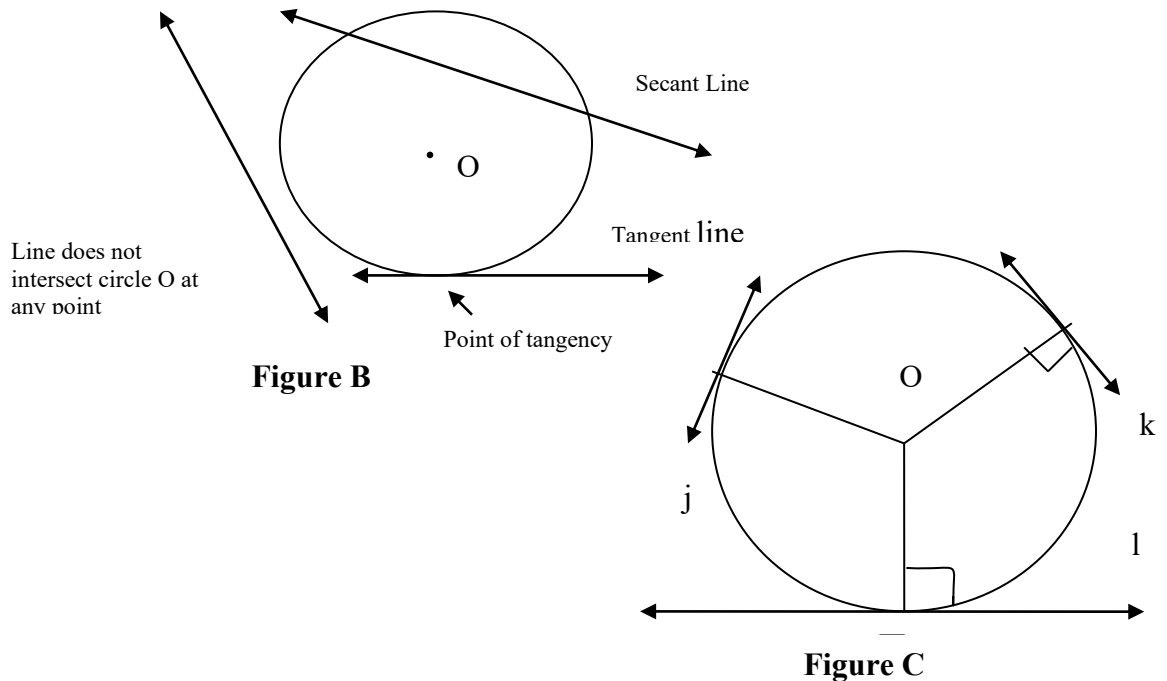
Examples:

- In figure A, angle AOB is a central angle.
- In figure A, AB is an intercepted arc.

Definitions:

- A tangent line is a line that intersects a circle in exactly one point; the point of congruency.
- A secant line is a line that intersects the circle in two different points.

Circles, Angle Measurement and Associated Concepts



Examples:

- In figure B, it is illustrated that a line may intersect a given circle at no point, one point, or two points.
- In figure C, it is illustrated that a radius drawn to a point of tangency is perpendicular to the tangent line.

More Definitions:

A polygon is circumscribed about a circle if each of its sides is tangent to the circle.

A polygon is inscribed in a circle if each of its vertices lies on the circle.

An inscribed angle is an angle whose vertex lies on a circle and whose sides are chords (or secants) of the circle.

Circles, Angle Measurement and Associated Concepts

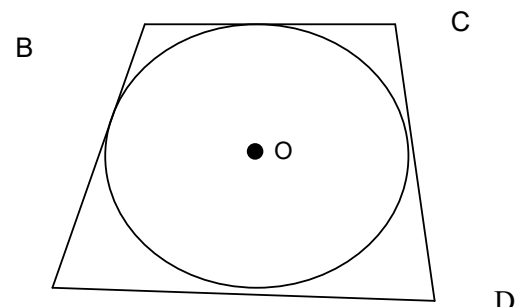


Figure D

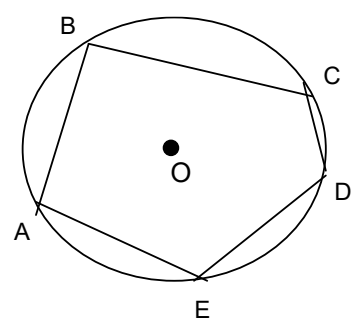


Figure E

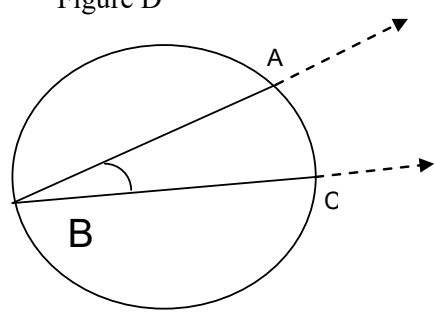


Figure F

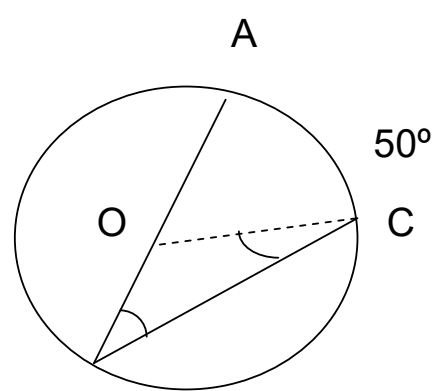


Figure G

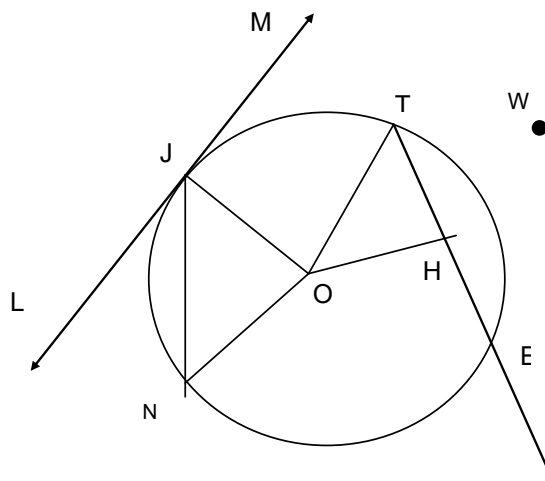
Examples:

- Figure D illustrates a circumscribed polygon.
- Figure E illustrates an inscribed polygon.
- Figure F illustrates an inscribed angle.

Remark: The measure of an inscribed angle is equal to one-half the measure of its intercepted arc.

Circles, Angle Measurement and Associated Concepts

Activity 1

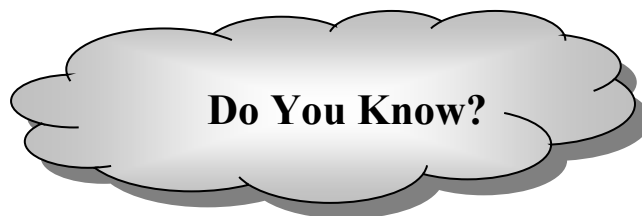


In Figure H,

1. Name the center of the circle _____
2. Name a radius of the circle _____
3. Name a chord of the circle _____
4. Name a diameter of the circle _____
5. Name a semicircle _____
6. Name a minor arc _____
7. Name a major arc _____
8. Name a tangent line _____
9. Name a secant line _____
10. Name an interior point _____
11. Name an exterior point _____
12. Name a point of tangency _____
13. Name a central angle _____
14. Name an inscribed angle _____
15. Name a right angle _____
16. Name the unit of measure for the circumference _____
17. Name the unit of measure of an arc _____
18. Name the unit of measure of an intercepted arc _____
19. Which is larger, an inscribed angle or a central angle? _____
20. What is the relationship between a tangent line to the circle and a radius to the point of tangency? _____

Theme I : The Foundation of Geometry

Lesson 5 : Area, Perimeter, Volume and Surface of Geometric Figures



Area

Any closed plane geometric figure has two dimensions or two different measures, usually length and width. When measuring a plane figure both measures must use the same linear units such as centimeters, meters, etc. When measuring the **area** of a plane, one measures the region inside the figure, and the measure of the area is expressed in square units such as square centimeters, square meters, etc. In summary, the **area** of a plane figure is the number of square units inside the region of the plane figure.

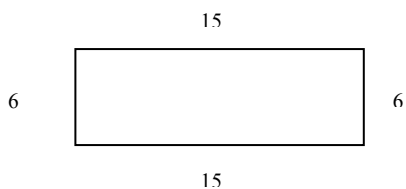
Perimeter

The **perimeter** of a closed geometric plane figure is the total distance around the boundary or outside edges of the figure. Its measure is expressed in a single linear measure.

Examples:

Area (A) and Perimeter (P)

Rectangle



$$\begin{aligned} A &= L \times W \text{ (length } \times \text{ width)} & P &= L + W + L + W \\ A &= 15 \times 6 & P &= 2L + 2W \\ A &= 90 \text{ sq. units} & P &= 15 + 6 + 15 + 6 \\ & & P &= 42 \text{ units} \end{aligned}$$

Volume

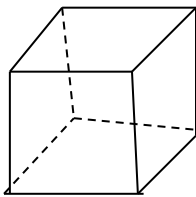
A closed geometric figure in three (3) space has three (3) measures, usually length, width and height; all three using the same linear unit. The **volume** of a closed geometric figure in three (3) space is expressed in cubic units (length, width, height) and it represents the number of cubic units inside the geometric figure.

Area, Perimeter, Volume and Surface of Geometric Figures

Surface

The **surface** of a three (3) dimensional closed geometric figure is the sum of the exterior (outer) measure of the boundary of the three (3) dimensional figure under consideration. The measure of the surface area of the three dimensional figure is expressed in square units.

Example: Volume of a cube and of a rectangular prism
 Volume = length x width x height ($V = lwh$)
 Surface area = sum of all boundary areas



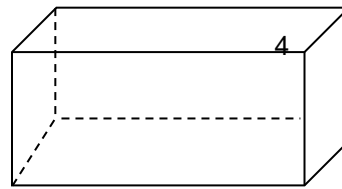
3

$$V = 3 \times 3 \times 3$$

$$V = 27$$

$$\text{Surface area} = 9+9+9+9 +9+9$$

$$= 54 \text{ square units}$$



8

$$V = 8 \times 4 \times 5$$

$$V = 160$$

$$\text{Surface area} = (4 \times 5) + (4 \times 5) + (4 \times 8)$$

$$+ (4 \times 8) + (5 \times 8) + (5 \times 8)$$

$$= 20+20+32+32+40+40$$

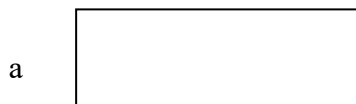
$$= 184 \text{ square units}$$

Geometric Formulas for Calculating Area (A) and Perimeter (P) of Plane Figures

Rectangle of Length b and width a

$$\text{Area} = ab$$

$$\text{Perimeter} = 2a + 2b$$



b
Figure A

Area of a Square

A Square is a rectangle with length and width being the same.

$$\text{Area} = s^2 \text{ (sides x sides)}$$



Figure B

Area, Perimeter, Volume and Surface of Geometric Figures

Parallelogram of Altitude h and Base b

$$\text{Area} = bh = ab \sin \theta$$

$$\text{Perimeter} = 2a + 2b$$

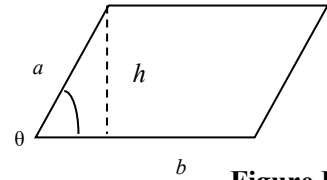


Figure B

Triangle of Altitude h and Base b

$$\begin{aligned} \text{Area} &= \frac{1}{2}bh = \frac{1}{2}ab \sin \theta \\ &= \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$

$$\text{where } s = \frac{1}{2}(a + b + c) = \text{semiperimeter}$$

$$\text{Perimeter} = a + b + c$$

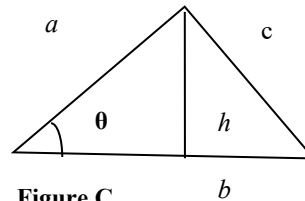


Figure C

Trapezoid of Altitude h and Parallel sides a and b

$$\text{Area} = \frac{1}{2}h(a + b)$$

$$\begin{aligned} \text{Perimeter} &= a + b + h \left(\frac{1}{\sin \theta} + \frac{1}{\sin \phi} \right) \\ &= a + b + h(\csc \theta + \csc \phi) \end{aligned}$$

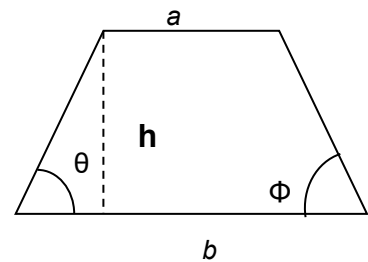
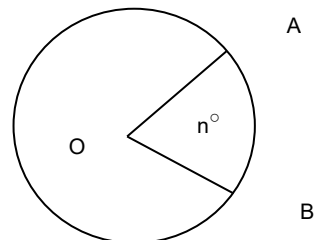


Figure D

Circles

A **sector** of a circle is a region of a circle bounded by two radii and the minor arc determined by these radii.

Example: Sector of a circle



Area, Perimeter, Volume and Surface of Geometric Figures

More Formulas for Plane Figures

Circle of Radius r

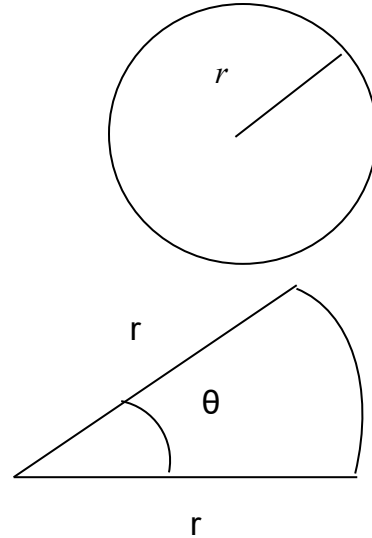
$$\text{Area} = \pi r^2$$

$$\text{Perimeter} = 2\pi r$$

Sector of Circle of Radius r

$$\text{Area} = \frac{1}{2}r^2\theta \quad [\theta \text{ in radians}]$$

$$\text{Arc length } s = r\theta$$



Geometric Formulas for Calculating Volume (V) and Surface Area (S)

Rectangular Prism Length a , Height b , width c

$$\text{Volume} = abc$$

$$\text{Surface area} = 2(ab+ac+bc)$$

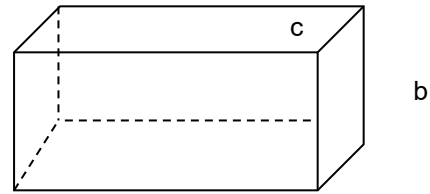


Figure A

Volume of a Cube

A cube is a rectangular prism with all faces equal, the same width and length.

$$\text{Volume} = \text{length} \times \text{width} \times \text{height} \quad (V=lwh)$$

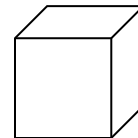


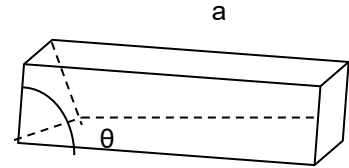
Figure B

Area, Perimeter, Volume and Surface of Geometric Figures

Volume of a Rectangular Prism of Cross-Sectional Area A and Height h

$$\text{Volume} = Ah = acb \sin \theta$$

Figure C



Sphere of Radius r

$$\text{Volume} = \frac{4}{3}\pi r^3$$

$$\text{Surface area} = 4\pi r^2$$

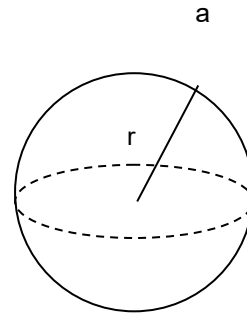


Figure D

Right Circular Cylinder of Radius r and Height h

$$\text{Volume} = \pi r^2 h$$

$$\text{Lateral surface area} = 2\pi r h$$

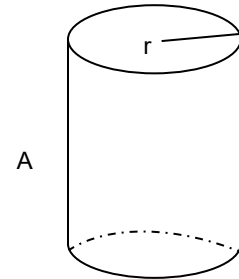


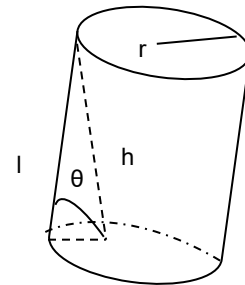
Figure E

Circular Cylinder of Radius r and Slant Height l

$$\text{Volume} = \pi r^2 h = \pi r^2 l \sin \theta$$

$$\text{Lateral surface area} = 2\pi r l = \frac{2\pi r h}{\sin \theta} = 2\pi r h \csc \theta$$

Figure F



Area, Perimeter, Volume and Surface of Geometric Figures

Right Circular Cone of Radius r and Height H

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

$$\text{Lateral surface area} = \pi r \sqrt{r^2 + h} = \pi r l$$

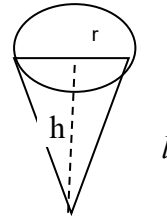


Figure G

Pyramid of Base Area A and Height h

$$\text{Volume} = \frac{1}{3}Ah$$

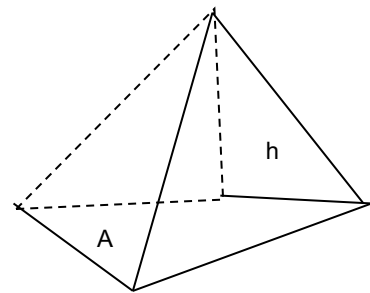


Figure I

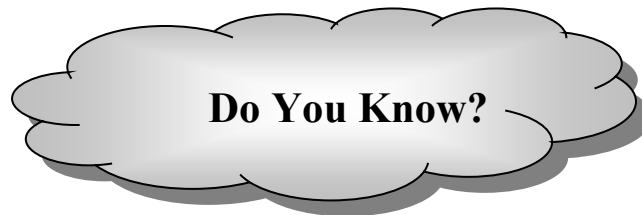
Area, Perimeter, Volume and Surface of Geometric Figures

Activity 1

- A. Find the area of each.
1. Rectangle: $l = 4.5$ $w = 8.1$
 2. Square: $\text{side} = 5$
 3. Parallelogram: $b = 14$ $h = 7$
 4. Triangle: $b = 10$ $h = 14$
- B. Solve for the volume.
5. Cube: $\text{side} = 16$
 6. Rectangular prism: $\text{length} = 4.2$, $\text{width} = 3$, $\text{height} = 8$
 7. Pyramid: $\text{length} = 4$, $\text{width} = 2$, $\text{height} = 6$
 8. Triangular Prism: $\text{base of triangle} = 2$
 $\text{height of triangle} = 5$
 $\text{height of prism} = 8$
 9. What is the perimeter of the figure in problem 1?
 10. What is the perimeter of the figure in problem 2?
 11. What is the area of a circle whose radius is 5?
 12. What is the surface area for the figure in problem 5?
 13. What is the surface area for the figure in problem 6?
 14. What is the perimeter of the circle in problem 11?
 15. Draw each of the figures in problems 1-14 and label them with the numbers given in the problems.

Theme II: Some Special Features of Geometry and Applications

Lesson 1: Symmetry, Similarity and Congruency




We call geometric figures that are flat and can be correctly drawn on a sheet of paper or a flat surface, a **two-dimensional figure or plane figure**. Many special two-dimensional figures can be drawn using line segments. Many special two dimensional geometric figures have angles. One well-known plane figure without angles is called a circle. Many plane figures have lines of symmetry. Several of these special two dimensional figures you should know by name. For more information about these, see Theme I, Lesson 5 in this book. A **line of symmetry** divides a plane figure into two equal figures. There are other figures in the plane that are not equal but they have the same basic shape. These figures are called **similar figures**. And then there are **congruent figures**, plane figures that are similar and equal in size.

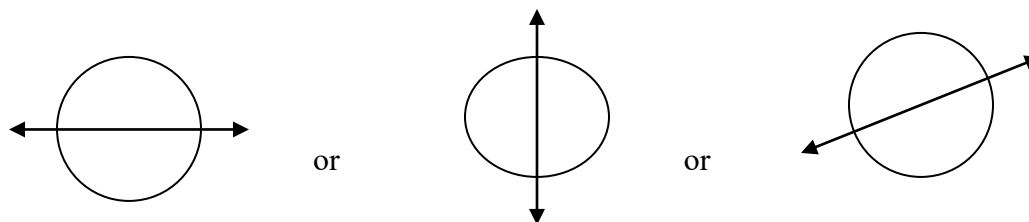
We call geometric figures that have volume and can be correctly drawn as figures with length, width, and height, a **three-dimensional figure or solid figure or a figure in 3-space**. Many special three-dimensional figures can be drawn using **vertices** (points), edges (line segments), and **faces** (plane figures) as part of their surface. Many special three-dimensional geometric figures have angles. One well-known three-dimensional or solid figure without angles is called a sphere. Many three-dimensional figures have **planes of symmetry**. Several of these special three-dimensional figures you should know by name. For more information about these, see Theme I, Lesson 5 in this book. A plane of symmetry divides a three-dimensional figure into two **congruent** (equal) figures. There are other geometric figures in 3-space that are not equal but they have the same basic shape. These figures are called similar figures. And then there are congruent figures, three-dimensional figures that are similar and equal. In this lesson we will introduce and study properties of symmetric figures, similar figures, and congruent figures.

Symmetry, Similarity and Congruency

Definition: A figure is said to have a **line of symmetry**, l , if the line l can be drawn in a way that divides the figure into two equal geometric figures.

Examples, consider a circle C . 

A line of symmetry is a line that passes through the diameter of the circle.

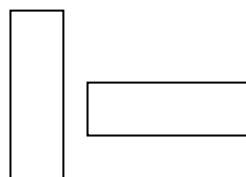


Definition: Two figures are said to be **similar** if they have the same shape, they may or may not have the same size.

Examples:



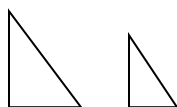
Not similar



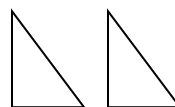
Similar

Definition: If two figures are similar and they have the same size, they are said to be **congruent**.

Examples:



Similar but not congruent

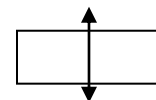


Similar and congruent

Properties (P)

P1. A figure may have more than one line of symmetry.

Example: A rectangle with two different lines of symmetry 



P2. A circle has only one distinct line of symmetry because a circle is viewed as having only one diameter.

P3. All similar figures are not congruent.

P4. All congruent figures are similar.

P5. A line of symmetry for a given figure divides that figure into different congruent figures.

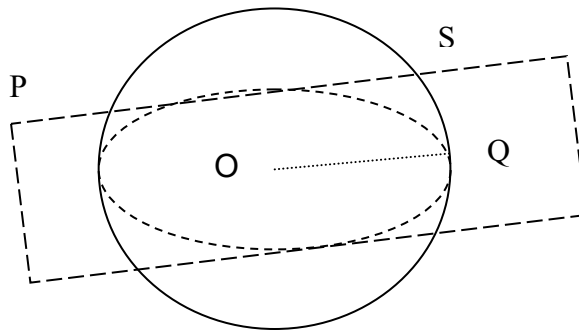
Symmetry, Similarity and Congruency

Plane of Symmetry, Similar and Congruent Solid Figures

Based on definitions given on Page 1 of this lesson, it is clear that the concepts of symmetry, similarity, and congruency for three-dimensional figures are natural extensions of the same concepts for two dimensional figures. Thus we will not discuss these concepts in depth for three-dimensional figures.

Definitions: Every circle in three space is contained in a unique plane in two-space; a circle on a sphere whose unique plane passes through the center of the sphere is called a great circle; all other circles are small circles.

For a sphere in three-space, the only planes of symmetry are those containing a great circle.



Example:

The plane contains a great circle and is a plane of symmetry.

Remarks: since any two great circles cannot really be distinguished, one from the other, we think of a sphere as having only one plane,. Some geometric solids have more than one plane of symmetry. Consider the regular hexagonal prism shown below.

Example: Two planes of symmetry for the regular hexagonal prism in Figure A.

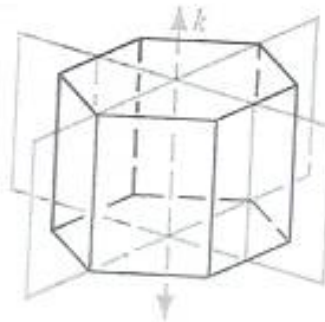
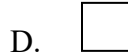
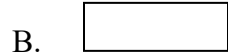
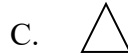
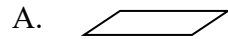
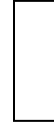


Figure A:

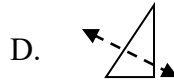
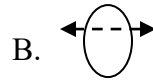
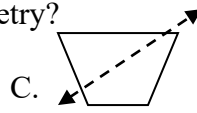
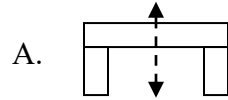
Symmetry, Similarity and Congruency

Activity 1:

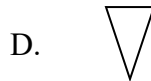
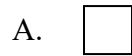
1. Which figure below is similar to this quadrilateral?



2. Which line is a line of symmetry?



3. Which figure has no lines of symmetry?

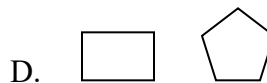
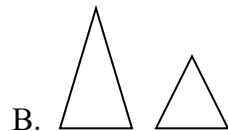
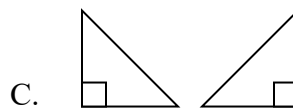
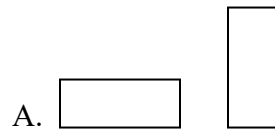


4. What would be a line of symmetry for

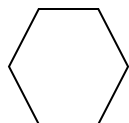
S

W

5. Which pairs of figures appear to be similar, congruent or neither?



6. How many different **lines of symmetry** can you draw for these figures?



Symmetry, Similarity and Congruency

Activity 2:

1. A regular hexagonal prism has several distinct planes of symmetry; identify the other planes of symmetry in figure A.
 2. A cube has how many distinct planes of symmetry? _____ Draw a cube and illustrate these.
 3. For a rectangular prism, how many distinct planes of symmetry can you determine? _____ Draw and illustrate these.
 4. For a right circular cylinder, how many distinct planes of symmetry can you determine? _____ Draw and illustrate these.
 5. For a right circular cone of radius r and _____??_ how many distinct lines of symmetry can you determine? _____ Draw and illustrate these.
- * For exercises 6-10, True (1) or False (2)
6. Every two cubes are similar.
 7. Every two cubes are congruent.
 8. In three-dimensional space, only solid identical shapes are similar.
 9. In three-dimensional space only solids that are identical in shape and size are congruent.
 10. In two-dimensional space, congruent plane figures do not have to be identical in size.

Theme II: Some Special Features of Geometry and Applications

Lesson 2: Proofs and Methods of Proof; Establishing Congruency of Triangles

Do You Know?

A. PROOFS

A **Theorem** in geometry (or mathematics in general) is a proposition or declarative statement that one can prove to be true; it is usually a generalization about an “entire set of things” rather than proving something about one a specific entity or thing.

The writing of a sequential series of short statements and reasons (explanations), beginning with what is given and concluding with what is desired to be proven is called a **Proof**. Most proofs use given information, definitions, postulates (or axioms) – including those from algebra and other branches of mathematics, and theorems that have already been proven as reasons or explanations for each the series of short statements listed in the left hand column. There is nothing mysterious about writing proofs; however, providing a sufficient and acceptable reason or explanation for each short sequential statement can sometime be challenging. The art of proving theorems is a process that one learns from experience.

There are no precise guidelines and there is no one way to prove a theorem. However, there are a few guidelines that have proven helpful to most persons who engage in proving theorems, namely:

- A. Understanding the facts and properties about aspects of the theorem to be proved.
- B. Understanding previous geometric (mathematical) results that have been established about aspects of geometry (mathematics) that relates to this theorem - what is already known?.
- C. Combining A and B together – observing how to best use each item of known information.

Most formal proofs in geometry are presented in a **Two-Column Format Proof**.

In the two column format, a series of statements or conclusions are listed in a column on the left with reasons (which justify each statement) in a column on the right. It should be noted that any two-column proof can be easily converted to a paragraph format proof by writing down each of the steps in the two column proof in the order they are presented and placing the reason after each step. It should be noted that a paragraph proof is really the goal, and that the objective of any proof is really to give an explanation as to why something is true, based on given information, definitions, postulates (axioms), and

Methods of Proof; Establishing Congruency of Triangles

previously established results. For most students, at all levels, success in writing proofs is a gradual process; it improves with experience.

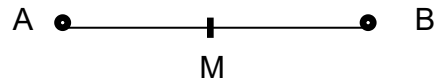
Example of a Two-Column Proof

Theorem Midpoint Theorem

If M is the midpoint of \overline{AB} , then $AM = \frac{1}{2} AB$ and $MB = \frac{1}{2} AB$.

Given: M is the midpoint of \overline{AB} .

Prove: $AM = \frac{1}{2} AB$; $MB = \frac{1}{2} AB$



Proof:

Statements	Reasons
1. M is the midpoint of \overline{AB} .	1. Given
2. $\overline{AM} \cong \overline{MB}$, or $AM = MB$	2. Definition of midpoint
3. $AM + MB = AB$	3. Segment Addition Postulate
4. $AM + AM = AB$, or $2AM = AB$	4. Substitution Prop. (Steps 2 and 3)
5. $AM = \frac{1}{2} AB$	5. Division Prop. of =
6. $MB = \frac{1}{2} AB$	6. Substitution Prop. (Steps 2 and 5)

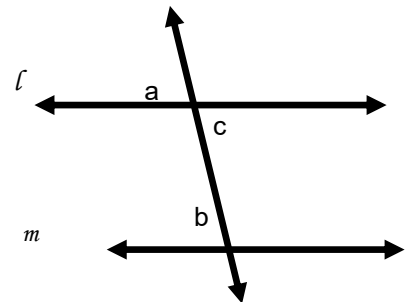
Theorem: If two parallel lines are cut by a transversal then their corresponding angles are congruent.

Given: $\ell \parallel m$.

Prove: $\angle a \cong \angle b$.

Solution

Plan: Introduce $\angle c$ in the diagram so that angles b and c are alternate interior angles. Then use Postulate 4.1 and vertical angles to establish the conclusion.



Proof:

Statements	Reasons
1. $\ell \parallel m$.	1. Given.
2. $\angle a \cong \angle c$.	2. Vertical angles are congruent.
3. $\angle c \cong \angle b$.	3. If two lines are parallel, then their alternate interior angles are congruent.
4. $\angle a \cong \angle b$	4. Transitive property.

Methods of Proof; Establishing Congruency of Triangles

Note: The notion of including a formal statement of a plan of attack was introduced. An indication of a “plan” and the corresponding annotation of the diagram is suggested before you begin to write the statements and reasons in the two-column proof.

B. Methods of Proof

In geometry (or mathematics in general) there are two (2) general kinds of proofs: (1) Direct Proofs and (2) Indirect Proofs.

Direct Proofs

Direct proofs use primarily deductive reasoning; beginning with what is given or assumed and using a series of short statements (justified by accepted facts – undefined terms, defined terms, postulates and previously proven theorems) to reason in a step-by-step fashion until a desired conclusion is reached.

Most direct proofs prove theorems in the “If . . . then . . .” form.

Example

If M is the midpoint of the line segment \overline{AB} , then $\overline{AM} = \frac{1}{2} AB$ and $\overline{MB} = \frac{1}{2} AB$

The information in the “If clause . . .” represents the hypothesis or the given part of the theorem is information that we are permitted to assume to be true at the beginning of the proof. The information in the “then clause . . .” represents what has to be proved.

The two (2) theorems presented earlier in this lesson are examples of direct proofs, beginning with theorems in the “If . . . then . . .” form.

There are three other statements related to the “If . . . then . . .” conditional proposition or theorem. They are listed below:

Given statement:	If p , then q .
Contrapositive:	If not q , then not p .
Converse:	If q , then p .
Inverse:	If not p , then not q .

A statement and its contrapositive are logically equivalent.

A statement is not logically equivalent to its converse or to its inverse.

There are other forms of direct proofs. However, all are similar to the way an “If . . . then . . .” statement is proven.

Methods of Proof; Establishing Congruency of Triangles

Indirect Proof

Since a conditional “If . . . then . . .” statement and its contrapositive are logically equivalent, one may prove the conditional statement by its contrapositive. This is an indirect method of proof.

In general the indirect methods of proof in geometry (mathematics) are strategies to prove a statement indirectly. To do this, one should first account for all logical outcomes or possibilities. Then one should attempt to prove each of these assumed possibilities (except the one to be proved) and show that such attempts all head to a contradiction or an impossibility. One then conclude that the remaining possibility (the desired conclusion) must be true. In summary, the indirect method of proof follows three (3) fundamental steps:

- Step 1: Assume that the information in the “then . . . (conclusion) clause” is not true.
- Step 2: Show that the “If . . . (given) clause” and the changed “then . . . (conclusion) clause contradicts a known fact.
- Step 3: Conclude that the original “If . . . then . . .” statement is true.

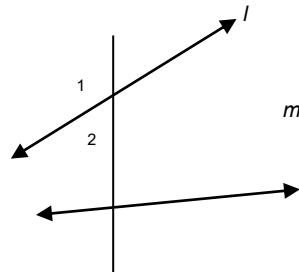
Example

Use the indirect method of proof to establish the theorem.

Theorem

If angle 1 is not congruent to angle 2, then line l is not parallel to line m .

Given: $\angle 1$ is not congruent to $\angle 2$
Prove: Line l is not parallel to line m .



Solution

Use an indirect method of proof:

1. There are two possibilities; the original statement and its negation: line l is parallel to line m .
2. Assume the negation is true; that is, line l is parallel to line m .
3. If the lines are parallel, then corresponding angles are congruent, so $\angle 1 \cong \angle 2$. But this contradicts the Given.
4. Since its negation is false, the statement “line l is not parallel to line m ” must be true since it is the only remaining possibility.

Methods of Proof; Establishing Congruency of Triangles

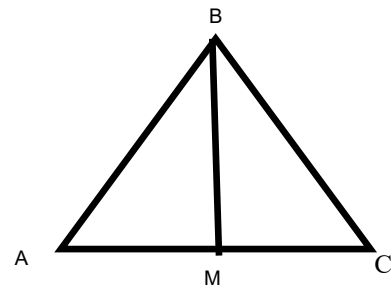
C. Proving Triangles are Congruent: SSS, SAS, ASA Theorems (Postulates)

Two triangles that are exactly the same if we could place one on top of the other are said to be congruent. In that a great deal of work in geometry is devoted to trying to prove that two (2) triangles are congruent (equal), we will conclude this lesson with three proofs establishing the congruency of two triangles. Some persons prove these as theorems, others accept them as postulates.

Theorem (Postulate) SSS: If the vertices of two triangles can be paired so that three sides of one triangle are congruent to the corresponding sides of the second triangle, then the two triangles are congruent.

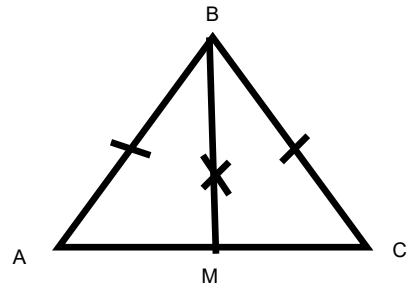
Proof

Given: $\overline{AB} \cong \overline{BC}$,
 M is the midpoint of \overline{AC} .
 Prove: $\triangle AMB \cong \triangle CMB$



Solution

- Plan:
1. Mark the diagram with the Given. Note that an “x” is used to indicate that side BM is congruent to itself
 2. After the three pairs of congruent sides are identified, write the formal proof.



Proof:

Statements	Reasons
1. $\overline{AB} \cong \overline{BC}$. (Side)	1. Given.
2. M is the midpoint of \overline{AC} .	2. Given.
3. $\overline{AM} \cong \overline{CM}$. (Side)	3. A midpoint divides a segment into two congruent segments.
4. $\overline{BM} \cong \overline{BM}$. (Side)	4. Reflexive property of congruence.
5. $\triangle AMB \cong \triangle CMB$	5. SSS Postulate.

Methods of Proof; Establishing Congruency of Triangles

Theorem (Postulate) ASA: If the vertices of two triangles can be paired so that two angles and the included side of one triangle are congruent to the corresponding parts of the second, then the two triangles are congruent.

Proof

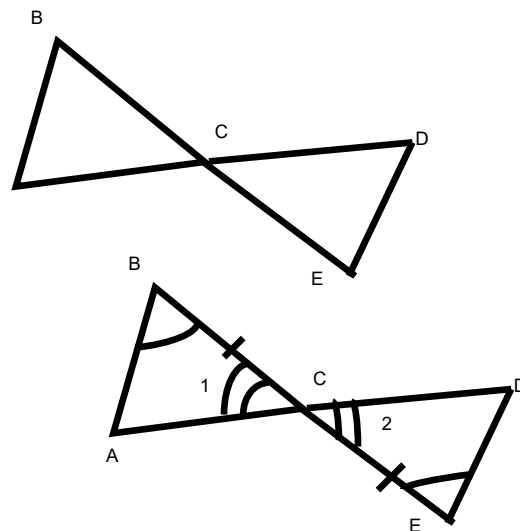
Given: C is the midpoint of \overline{BE} ,

$$\angle B \cong \angle E.$$

Prove: $\triangle ABC \cong \triangle DEC$.

Solution

- Plan:
1. Mark the diagram with the Given and any additional parts, such as the vertical angle pair, that can be deduced to be congruent based on the diagram.
 2. Examine the diagram marked in Step 1 to decide which method of proving triangles congruent to use. In this case, use ASA.
 3. Write the formal proof.



Proof:

Statements

Reasons

1. $\angle B \cong \angle E$. (Angle)

1. Given.

2. C is the midpoint of \overline{BE} .

2. Given.

3. $\overline{BC} \cong \overline{EC}$. (Side)

3. A midpoint divides a segment into two congruent segments.

4. $\angle 1 \cong \angle 2$. (Angle)

4. Vertical angles are congruent.

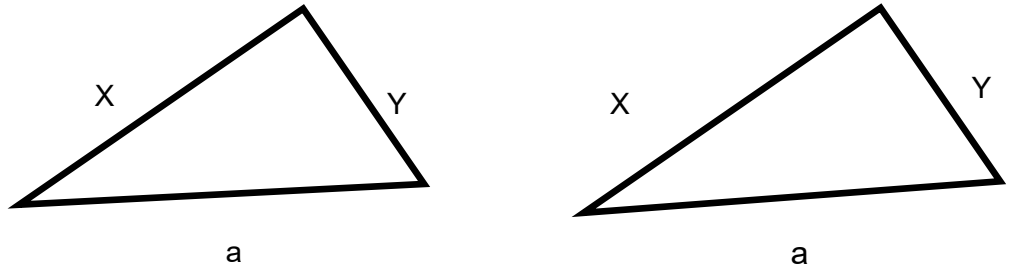
5. $\triangle ABC \cong \triangle DEC$.

5. ASA Postulate.

Methods of Proof; Establishing Congruency of Triangles

Theorem (Postulate) SAS: If two sides and the included angle of one triangle are congruent to the corresponding parts of a second triangle, then the two triangles are congruent.

Proof



Side Angle Side (SAS)

Proof

Assume the angle of a triangle, and the adjacent sides that form that angle are congruent to the angle and adjacent sides of a second triangle. Move the first triangle onto the second so that side angle side lines up with side angle side. If the left sides line up, and the right sides don't, one angle is inside the other, hence smaller than the other. Since the angles are congruent, both pairs of sides line up. Does the vertex at the end of the left side of the first triangle coincide with the vertex at the end of the left side of the second triangle? Both are a fixed distance from the common apex, along a common line, hence they are the same point. By the same reasoning, the third vertex of the first triangle coincides with the third vertex of the second triangle. The sides and angles all coincide and the triangles are congruent. This method of proving congruence is called SAS.

Definition: A **Proposition** is a declarative statement that can be proven to be true or false.

Activity 1

For each Theorem written in this lesson, state the following:

The Contrapositive

The Converse

The Inverse

And attempt to prove at least one of these propositions associated with each Theorem.

Theme II : Some Special Features of Geometry and Applications

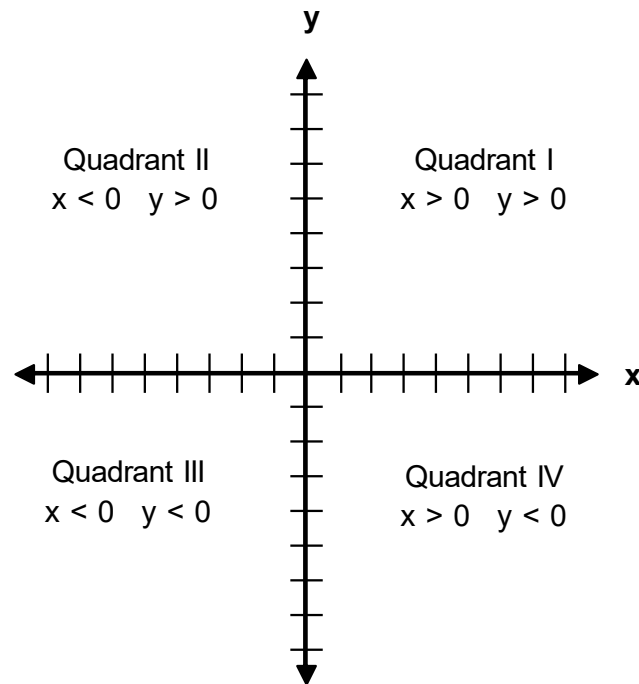
Lesson 3: The Coordinate Geometry and Transformational Geometry

Do You Know?

COORDINATE GEOMETRY

A flat surface extending to infinity in all directions is a representation of a geometric $x - y$ plane. The x -axis and the y -axis divide a plane into four regions called quadrants. The quadrant where both x and y are positive is called quadrant I. The other three quadrants are labeled below. In geometry, two distinct lines determine a unique plane. The x and y axes determine a unique $x - y$ plane. Every point A in the $x - y$ coordinate plane is associated with an ordered pair of real numbers (x, y) called the coordinates of A , i.e.

$A(x, y)$. Working with geometric expressions and figures in the $x - y$ plane where every point is associated with an ordered pair of points (x, y) is called **Coordinate Geometry**..



The Coordinate Geometry and Transformational Geometry

This $x - y$ -coordinate system identified here is usually called the rectangular or Cartesian coordinate system. (The Cartesian system is named after the French philosopher and mathematician Descartes.) There are other types of coordinate systems that are sometimes used in geometry

x-axis: Usually, the horizontal axis of the coordinate plane. Positive positions are measured to the right of the origin; negative positions, to the left.

y-axis: Usually, the vertical axis of the coordinate plane. Positive positions are measured up from the origin; negative positions, down from the origin.

Origin: $(0,0)$, the point of intersection of the x -axis and the y -axis; these two axes are perpendicular at the origin. .

Quadrants: The four regions of the plane cut by the two axes. By convention, they are numbered counterclockwise starting with the upper right.

Point: A location on a plane identified by an ordered pair of coordinates enclosed in parentheses. The first coordinate is measured along the x -axis; the second, along the y -axis. Ex: The point $(1, 2)$ is 1 unit to the right and 2 units up from the origin.

A straight line is uniquely identified by any two points, or by any one point and the incline, or slope, of the line.

The **slope** of a line in the Cartesian plane measures how steep it is; a ratio of x distance over y distance.

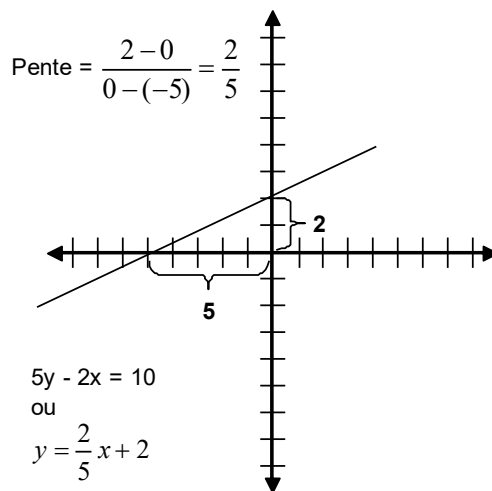
A Linear Equations is the equation of a straight line..

Standard Form: $Ax + By = C$ of a linear equation.

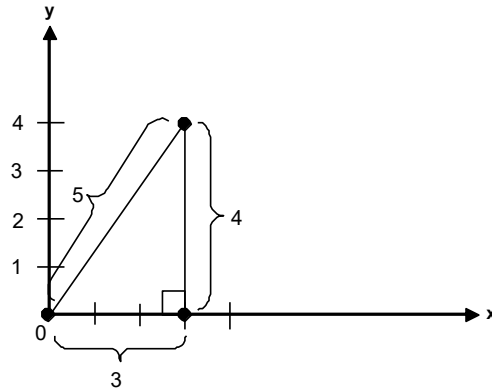
$$\text{Slope} = \frac{A}{B}; \text{ y-intercept} = \frac{C}{B}; \text{ x-intercept} = \frac{C}{A}.$$

Slope-intercept form: $y = mx + b$ or : $y - k = m(x - h)$ of a linear equation; where m is the slope, (h,k) is a point on the line;

The graph of the equation of a straight line in coordinate geometry.



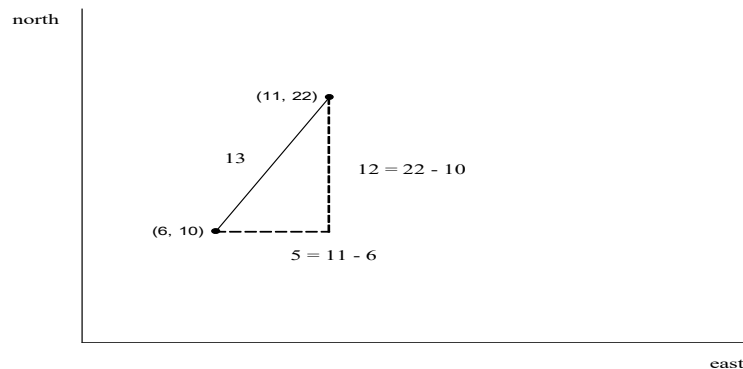
The Coordinate Geometry and Transformational Geometry



When two points do not lie on a horizontal or vertical line, you can find the distance between the points by using the Pythagorean Theorem.

Example Find the distance between points A(4, -2) and B(1, 2).

Solution Draw the horizontal and vertical segment shown. The coordinates of T are (1, -2). Then AT = 3, BT = 4, $(AB)^2 = 3^2 + 4^2 = 25$, and AB = 5.



Example The distance between two points A (6, 10) and B (11, 22) is

$$d = \sqrt{(11-6)^2 + (22-10)^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

The **standard form** for the formula for the **distance** between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

A general formula for the distance from the origin to the point (x, y) is $d = \sqrt{x^2 + y^2}$

The Coordinate Geometry and Transformational Geometry

TRANSFORMATIONAL GEOMETRY

Transformational geometry is the geometry in the x-y coordinate plane that permits points, sets of points and geometric objects to be moved from one position to another in some logical manner.

A **transformation** is a rule or mapping that assigns each point in the x-y coordinate plane to a different point or to itself.

In this lesson we present definitions and examples of four widely known and widely used transformations; Reflections, Translations, Rotations and Dilations.

A **reflection** in line m is a transformation of the plane having the property that the point O on m is O itself.

We write $R_m(S) = S'$ to mean S' is the image of S under the reflection in line m .

A **translation** is a transformation of the plane such that the image of every point (a, b) is the point $(a + h, b + k)$, where h and k are given.

A translation has the effect of moving every point the same distance in the same direction. We use the notation $T_{(h,k)}(a,b)$ to mean the image of (a, b) under a translation of h units in the x-direction and k units in the y-direction.

As in a reflection, distance and angle measure are preserved in a translation.

A **rotation** through an angle of measure θ degrees about a point P is a transformation of the plane such that the image of P is P and, for any point $B \neq P$, the image B is B' , where $m\angle BPB' = \theta$ and $\overline{BP} \cong \overline{B'P}$.

If $\theta > 0$, the rotation is counterclockwise. If $\theta < 0$, the rotation is clockwise.

Given a point P in the plane and a positive number n , a transformation of the plane having the following properties is called a dilation of n , and P is called the center of dilation: Point P is fixed, and for any point Q , the image of Q is the point Q' such that

$PQ' = (n)(PQ)$ and \overrightarrow{PQ} and $\overrightarrow{PQ'}$ are identical rays.

The point Q' is usually denoted $D_n(Q)$.

The Coordinate Geometry and Transformational Geometry

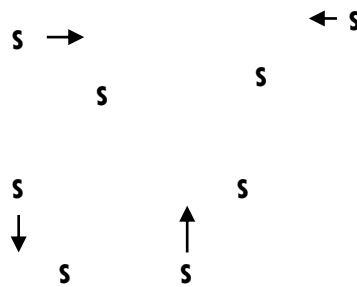
Several properties of dilations are known.

- Dilations do not preserve distance.
- The image of a figure is similar to the figure under a dilation.
- Angles are preserved under dilations.
- When the center of a dilation is $O = (0, 0)$, we can find the images of points very easily: $D_n(x, y) = (nx, ny)$.

Concisely speaking:

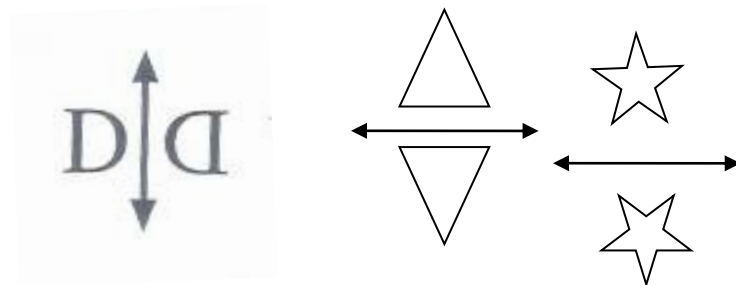
To **translate** a figure, move it right, left, up, down, or a combination of these.

Examples



To **reflect** a figure about a line, the figure is drawn backwards (like a mirror image).

Examples

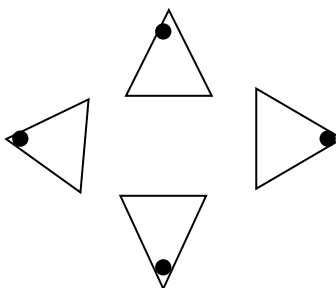


To **rotate** a figure, turn it in quarter turns like a clock.

(12:00)

Examples

$\frac{1}{4}$ counter-clockwise (9:00)



$\frac{1}{4}$ clockwise (3:00)

$\frac{1}{2}$ clockwise or counter-clockwise (6:00)

The Coordinate Geometry and Transformational Geometry

To **dilate** a geometric figure is to stretch or shrink it.

Example



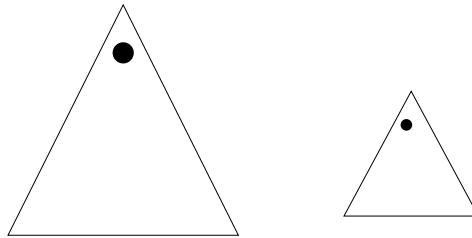
Stretch this figure by pulling two corners.



To **shrink** a figure, draw the same

figure, just smaller.

Example



Properties of Transformations

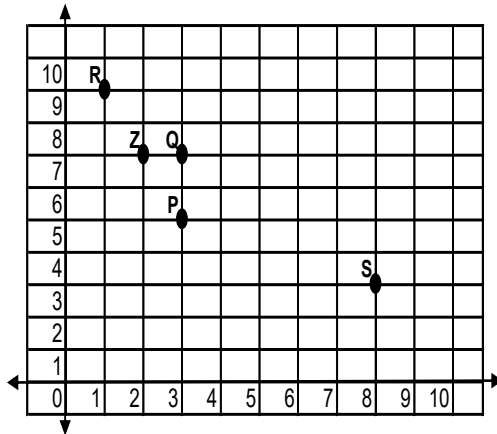
We are now in a position to summarize the properties of transformations. In particular, we are interested in what is preserved under each kind of transformation.

1. Reflections preserve (a) distance, (b) angle measure, (c) midpoints, (d) parallelism, and (e) colinearity.
2. Translations preserve these same five properties, (a) through (e).
3. Rotations preserve all five properties as well.
4. Dilations preserve all except distance, that is, (b) through (e).

The Coordinate Geometry and Transformational Geometry

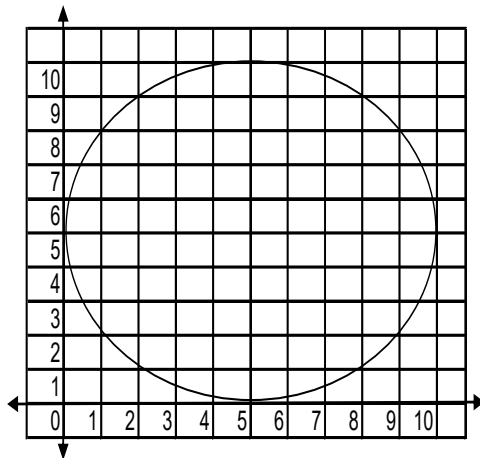
Activity 1

1. What are the coordinates of the point that are labeled on this coordinate plane?

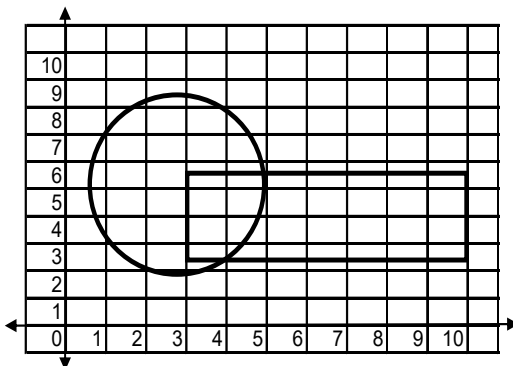


- Q ()
 P ()
 R ()
 S ()
 Z ()

2. Which ordered pair is closest to the center of this circle?



3. Name three ordered pairs that are :
- Inside the circle, but not on or inside the rectangle.
 - Inside the rectangle, but not on or inside the circle.
 - Inside both the circle and the rectangle.



4. Find the distance between the following pairs of points: $(-4, 6)$ and $(2, 10)$.
5. Find the equation of the line connecting the points: $(2, 3)$ and $(6, 10)$.

The Coordinate Geometry and Transformational Geometry

Activity 2

1. Draw an example of a reflection in the x-y coordinate plane using a line and a geometric figure.
2. Draw some examples of a translation in the x-y coordinate plane.
3. Draw some examples of a rotation in the x-y coordinate plane.
4. Draw some example of a dilation in the x-y coordinate plane.

Theme II : Some Special Features of Geometry and Applications

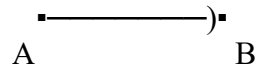
Lesson 4: Some Special Geometric Constructions

Do You Know?

In geometry one of the great challenges is to begin with a given figure and then construct a congruent one. Or to construct another geometric figure related to a given one. In this lesson geometric figures will be constructed using only two instruments, a **straight-edge** and a **compass**. Ten basic constructions will be presented congruent.

Construction 1: Given a segment, construct a segment congruent to the given segment.

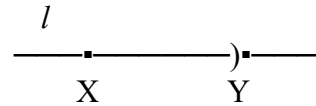
Given: \overline{AB}



Construct: A segment congruent to \overline{AB}

Procedure:

1. Use a straight-edge to draw a line. Call it l .
2. Choose any point on l and label it X.
3. Set your compass for radius AB. Using X as center, draw an arc intersecting line l . Label the point of intersection Y.



\overline{XY} is congruent to \overline{AB} .

Justification: Since you used AB for the radius of circle X, $\overline{XY} \cong \overline{AB}$.

Some Special Geometric Constructions

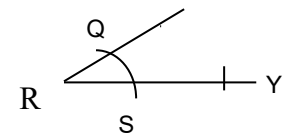
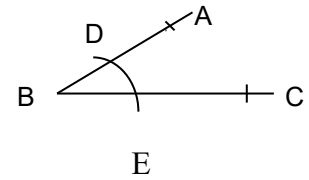
Construction 2: Given an angle, construct an angle congruent to the given angle.

Given: $\angle ABC$

Construct: An angle congruent to $\angle ABC$

Procedure:

1. Draw a ray. Label it \overrightarrow{RY} .
2. Using B as center and any radius, draw an arc intersecting \overrightarrow{BA} and \overrightarrow{BC} . Label the points of intersection D and E, respectively.
3. Using R as center and the same radius as in Step 2, draw an arc intersecting \overrightarrow{RY} . Label the arc \widehat{XS} , with S the point where the arc intersects \overrightarrow{RY} .
4. Using S as center and a radius equal to DE, draw an arc that intersects \widehat{XS} at a point Q.
5. Draw \overrightarrow{RQ} .



$\angle QRS$ is congruent to $\angle ABC$.

Justification: If you draw \overline{DE} and \overline{QS} , $\triangle DBE \cong \triangle QRS$ (SSS Postulate)

Construction 3: Given an angle, construct the bisector of the angle.

Given: $\angle ABC$

Construct: The bisector of $\angle ABC$

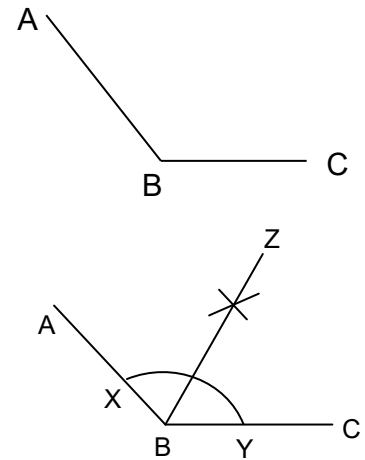
Procedure:

1. Using B as center and any radius, draw an arc that intersects \overrightarrow{BA} at X and \overrightarrow{BC} at Y.
2. Using X as center and a suitable radius, draw an arc. Using Y as center and the same radius, draw an arc that intersects the arc with center X at a point Z.
3. Draw \overrightarrow{BZ} .

\overrightarrow{BZ} bisects $\angle ABC$.

Justification: If you draw \overline{XZ} and \overline{YZ} , $\triangle XBZ \cong \triangle YBZ$ (SSS Postulate).

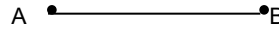
Then $\angle XBZ \cong \angle YBZ$ and \overrightarrow{BZ} bisects $\angle ABC$.



Some Special Geometric Constructions

Construction 4: Given a segment, construct the perpendicular bisector of the segment.

Given: \overline{AB}



Construct: The perpendicular bisector of \overline{AB}

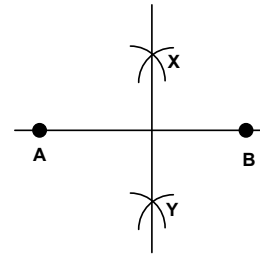
Procedure:

- Using any radius greater than $\frac{1}{2}AB$, draw four arcs of equal radii, two with center A and two with center B. Label the points of intersections of these arcs X and Y.

- Draw \overleftrightarrow{XY} .

\overleftrightarrow{XY} is the perpendicular bisector of \overline{AB} .

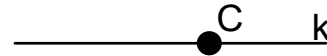
Justification: Points X and Y are equidistant from A and B. Thus \overleftrightarrow{XY} is the perpendicular bisector of \overline{AB} .



Remark: One can use Construction 4 to find the midpoint of a segment.

Construction 5: Given a point on a line, construct the perpendicular to the line at the given point.

Given: Point C on line k

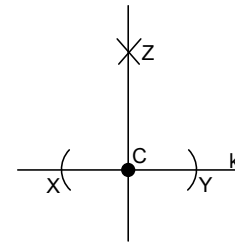


Construct: The perpendicular to k at C

Procedure:

- Using C as center and any radius, draw an arc intersecting k at X and Y.
- Using X as center and a radius greater than CX, draw an arc. Using Y as center and the same radius, draw an arc intersecting the arc with center X at a point Z.
- Draw \overleftrightarrow{CZ} .

\overleftrightarrow{CZ} is perpendicular to k at C.



Justification: You constructed points X and Y so that C is equidistant from X and Y. Then you constructed point Z so that Z is equidistant from X and Y. Thus \overleftrightarrow{CZ} is the perpendicular bisector of \overline{XY} , and $\overleftrightarrow{CZ} \perp k$ at C.

Some Special Geometric Constructions

Construction 6: Given a point outside a line, construct the perpendicular to the line from the given point.

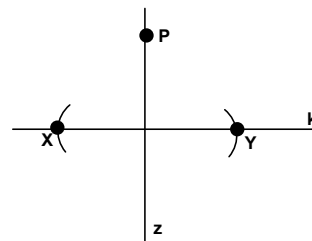
Given: Point P outside line k

Construct: The perpendicular to k from P



Procedure:

1. Using P as center, draw two arcs of equal radii that intersect k at points X and Y.
2. Using X and Y as centers and a suitable radius, draw arcs that intersect at a point Z.
3. Draw \overleftrightarrow{PZ} .



\overleftrightarrow{PZ} is perpendicular to k .

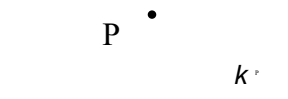
Justification: Both P and Z are equidistant from X and Y.

Thus \overleftrightarrow{PZ} is the perpendicular bisector of \overline{XY} , and $\overleftrightarrow{PZ} \perp k$.

Construction 7: Given a point outside a line, construct the parallel to the given line through the given point.

Given: Point P outside line k

Construct: The line through P parallel to k

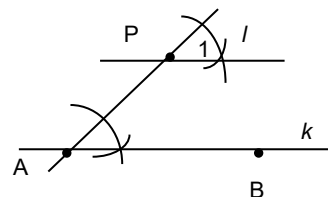


Procedure:

1. Let A and B be two points on line k . Draw \overleftrightarrow{PA} .
2. At P, construct $\angle 1$ so that $\angle 1$ and $\angle PAB$ are congruent corresponding angles. Let l be the line containing the ray you just constructed.

l is the line through P parallel to k .

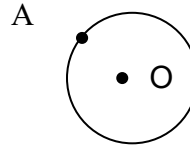
Justification: If two lines are cut by a transversal and corresponding angles are congruent, then the lines are parallel. (Postulate)



Some Special Geometric Constructions

Construction 8: Given a point on a circle, construct the tangent to the circle at the given point.

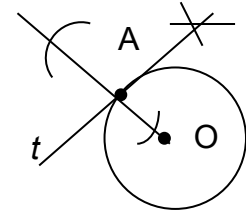
Given: Point A on circle O



Construct: The tangent to circle O at A

Procedure:

1. Draw \overrightarrow{OA} .
2. Construct the line perpendicular to \overrightarrow{OA} at A. Call it t .



Line t is tangent to circle O at A.

Justification: Because t is perpendicular to radius \overline{OA} at A, t is tangent to circle O.

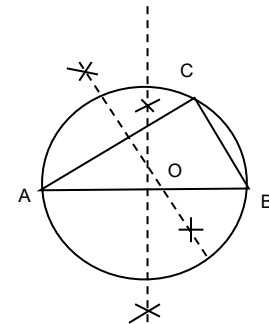
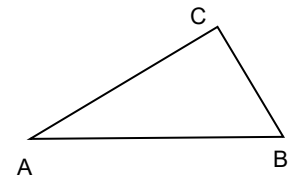
Construction 9: Given a triangle, circumscribe a circle about the triangle.

Given: $\triangle ABC$

Construct: A circle passing through A, B, and C

Procedure:

1. Construct the perpendicular bisector of any two sides of $\triangle ABC$. Label the point of intersection O.
2. Using O as center and OA as radius, draw a circle.



Circle O passes through A, B, and C.

Justification: Inscribed Triangle Property.

Some Special Geometric Constructions

Construction 10: Given a triangle, inscribe a circle in the triangle.

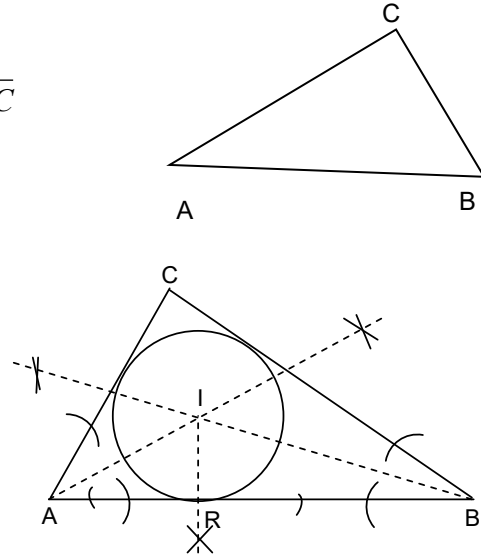
Given: $\triangle ABC$

Construct: A circle tangent to \overline{AB} , \overline{BC} , and \overline{AC}

Procedure:

1. Construct the bisector of $\angle A$ and $\angle B$.
Label the point of intersection I .
2. Construct a perpendicular from I to \overline{AB} ,
intersecting \overline{AB} at a point R .
3. Using I as center and IR as radius, draw
a circle.

Circle I is tangent to \overline{AB} , \overline{BC} , and \overline{AC} .

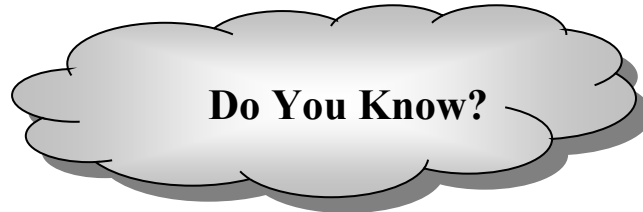


Activity 1

Get a straight-edge and compass and do each of the ten (10) constructions presented in this lesson.

Theme II : Some Special Features of Geometry and Applications

Lesson 5: Applications – Using Geometry to Solve Real World Problems



The word geometry is derived from the Greek words *geos* (meaning earth) and *metron* (meaning measure). The ancient Egyptians, Chinese, Babylonians, Romans, and Greeks used geometry for surveying, navigation, astronomy, and other practical occupations.

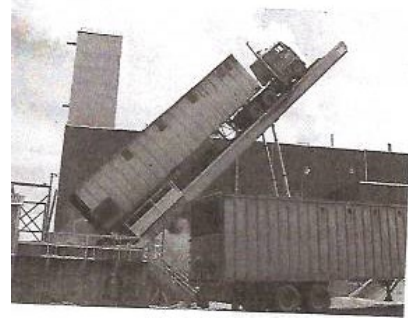
The Greeks sought to systematize the geometric facts they knew by establishing logical reasons for them and relationships among them. The work of such men as Thales (600 B.C.), Plato (390 B.C.), and Aristotle (350 B.C.) in systematizing geometric facts and principles culminate in the geometry text *Elements*, written about 325 B.C. by Euclid. This most remarkable text has been in use for over 2000 years.

Over the centuries, every culture and every country has used and studied geometry. Some have had unique approaches for doing this. However, most have developed or arrived at using geometry the way that Euclid and the Greeks did. Additionally, every culture and every country has made practical application of geometry. In this lesson, a variety of practical applications of geometry is presented.

Applications – Using Geometry to Solve Real World Problems

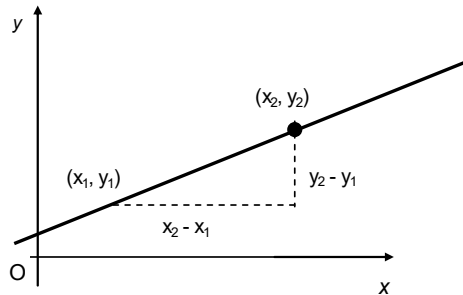
Example 1 Slope of a Line

The effect of steepness, or slope, must be considered in a variety of everyday situations. Some examples are the grade of a road, the pitch of a roof, the incline of a wheelchair ramp, and the tilt of an unloading platform, such as the one at the mill in Maine shown in the photograph at the right. In this example, the informal idea of steepness is generalized and made precise by the mathematical concept of slope of a line through two points.



Informally, slope is the ratio of the change in y (vertical change) to the change in x (horizontal change). The slope, denoted by m , of the nonvertical line through the points (x_1, y_1) and (x_2, y_2) is defined as follows:

$$\begin{aligned} \text{slope } m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\text{change in } y}{\text{change in } x} \end{aligned}$$



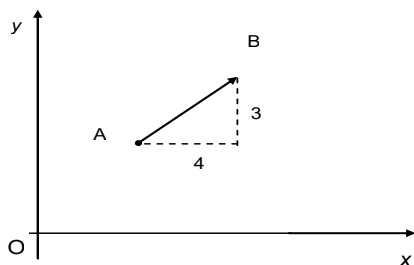
Example 2 Vectors

The journey of a boat or airplane can be described by giving its speed and direction, such as 50 km/h northeast. Any quantity such as force, velocity, or acceleration, that has both magnitude (size) and direction, is a **vector**.

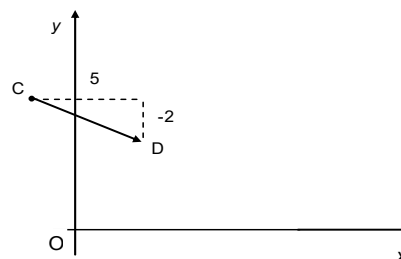


When a boat moves from point A to point B , its journey can be represented by drawing an arrow from A to B , \overline{AB} (read “vector AB ”). If \overline{AB} is drawn in the coordinate plane, then the journey can also be represented as an ordered pair.

$$\overline{AB} = (\text{change in } x, \text{change in } y)$$



$$\overline{AB} = (4, 3)$$



$$\overline{CD} = (5, -2)$$

Applications – Using Geometry to Solve Real World Problems

The **magnitude** of a vector \overline{AB} is the length of the arrow from point A to point B and is denoted by the symbol $|\overline{AB}|$. You can use the Pythagorean Theorem or the Distance Formula to find the magnitude of a vector. In the diagrams above,

$$|\overline{AB}| = \sqrt{4^2 + 3^2} = 5$$

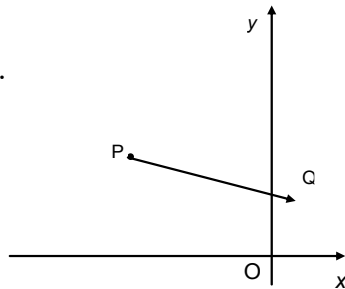
and

$$|\overline{CD}| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

Example Given: Points P(-5, 4) and Q(1, 2)

- Sketch \overline{PQ} .
- Find \overline{PQ} .
- Find $|\overline{PQ}|$.

Solution a.



b. $\overline{PQ} = (1 - (-5), 2 - 4) = (6, -2)$

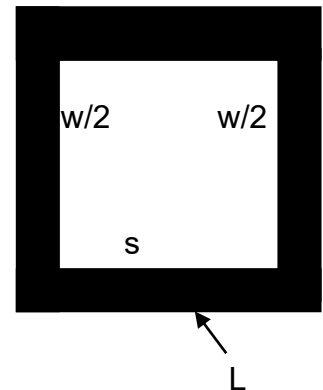
c. $|\overline{PQ}| = \sqrt{6^2 + (-2)^2} = \sqrt{40} = 2\sqrt{10}$

Example 3 Sidewalk

Show that the area of a sidewalk that borders a square (shown in the Figure) is simply its width (w) times the total length of its midline (L):

$$K = wL$$

(The **midline** is defined as the locus of midpoints of all line segments joining two adjacent borders of the shaded region.)



Solution

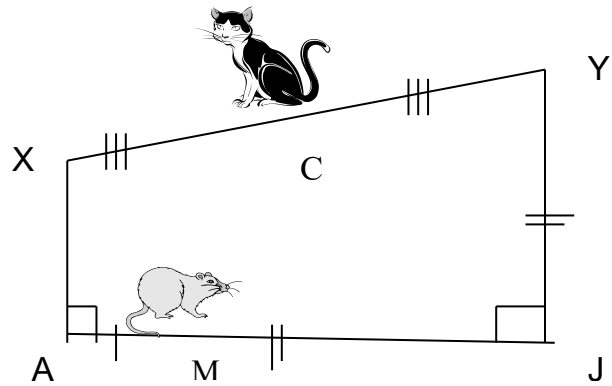
Let s = side of the inner square; then the side of the outer square will be $s + 2w$. The area of the sidewalk is the difference of the areas of these two squares, so

$$\begin{aligned} K &= (s + 2w)^2 - s^2 \\ &= s^2 + 4sw + 4w^2 - s^2 \\ &= 4sw + 4w^2 \end{aligned}$$

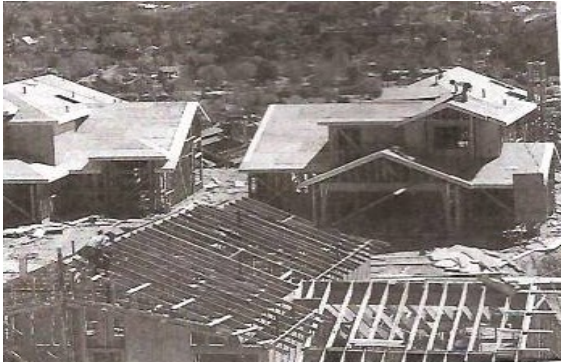
Applications – Using Geometry to Solve Real World Problems

Example 4 Cat and Mouse Game

A mouse moves along \overline{AJ} . For any position M of the mouse, X and Y are such that $\overline{AX} \perp \overline{AJ}$ with $AX = AM$, and $\overline{JY} \perp \overline{AJ}$ with $JY = JM$. The cat is at C , the midpoint of \overline{XY} . Describe the locus of the cat as the mouse moves from A to J .



Example 5

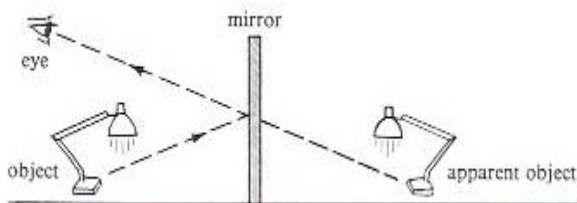
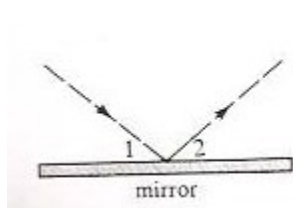


Although some carpenters learn the trade through four-year apprenticeships, most learn on the job. These workers begin as laborers or as carpenters' helpers. While they work in these jobs they gradually acquire the skills necessary to become carpenters themselves. Carpenters must be able to measure accurately and to apply their knowledge of arithmetic, geometry, and informal algebra. They also benefit from being able to read and understand plans, blueprints, and charts.

Example 6 : Mirrors

If a ray of light strikes a mirror at an angle of 40° , it will be reflected off the mirror at an angle of 40° also. The angle between the mirror and the reflected ray is always congruent to the angle between the mirror and the initial light ray. In the diagram at the left below, $\angle 2 \cong \angle 1$.

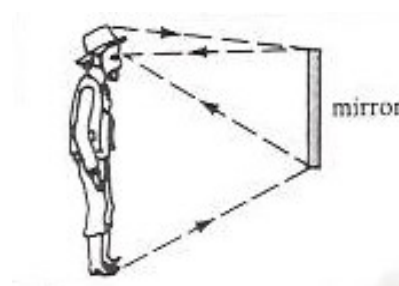
Applications – Using Geometry to Solve Real World Problems



Example 7 Mirror Objects

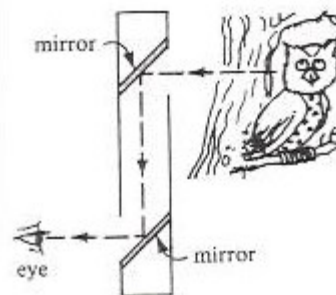
We see objects in a mirror when the reflected light ray reaches the eye. The object appears to lie behind the mirror as shown in the diagram at the right.

You don't need a full-length mirror to see all of yourself. A mirror that is only half as tall as you are will do if the mirror is in a position as shown. You see the top of your head at the top of the mirror and your feet at the bottom of the mirror. If the mirror is too high or too low, you will not see your entire body.



Example 8 The Geometry of a Periscope Construction

A periscope uses mirrors to enable a viewer to see above the line of sight. The diagram at the right is a simple illustration of the principle used in a periscope. It has two mirrors, parallel to each other, at the top and at the bottom. The mirrors are placed at an angle of 45° with the horizontal. Horizontal light rays from an object entering at the top are reflected down to the mirror at the bottom. They are then reflected to the eye of the viewer.



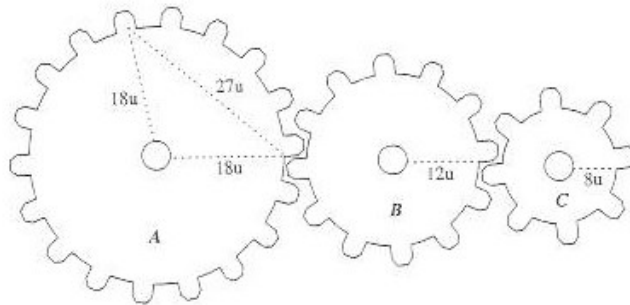
Applications – Using Geometry to Solve Real World Problems

Example 9 Designing and Operating Gears in Machines

Two gears are said to have **gear ratio** $a/b = k$ if the first gear has p teeth, the second gear has q teeth, and $p/q = k$. The gear ratio determines how fast the gears turn with respect to each other. For example, a gear ratio of $3/2 = 1.5$ means that the smaller gear will make 1.5 revolutions for every one revolution made by the larger gear. Three gears in tandem (as shown in Figure 4.32) are such that the relative turning speed of Gear A to Gear B equals that of Gear B to Gear C (which is itself undetermined). If Gear A has 18 teeth and Gear C had 8 teeth, the number of teeth in Gear B is the geometric mean of the numbers for gears A and C. This is because, using gear ratio as defined above,

$$\frac{18}{x} = \frac{x}{8}$$

or, by algebra, $x^2 = 18 \cdot 8 = 144$, which yields the answer $x = 12$ teeth for Gear B. The relative turning speeds can now be found: That of Gear C to Gear A is $18/8 = 2.25$, whereas those of Gear C to Gear B and of Gear B to Gear A (stated to be equal) are each equal to $18/12 = 1.25$. Note that $2.25 = (1.25)^2$. An interesting fact for gear trains in linear sequence (such this one) is that if there are $n + 1$ gears in tandem, and all engaged pairs have equal relative turning speeds $k > 1$, then the smaller gear has a turning speed of k^n with respect to the larger gear.



Applications – Using Geometry to Solve Real World Problems

Example 10 Rotations

A rotation is a transformation suggested by rotating a paddle wheel. When the wheel moves, each paddle rotates to a new position. When the wheel stops, the new position of a paddle (P') can be referred to mathematically as the image of the initial position (P).



For the counterclockwise rotation shown about point O through 90° , we write $\mathcal{R}_{O, 90}$. A counterclockwise rotation is considered positive, and a clockwise rotation is considered negative. If the red paddle is rotated about O clockwise until it moves into the position of the black paddle, the rotation is denoted by $\mathcal{R}_{O, -90}$. (Note that to avoid confusion with the R used for reflection we use a script \mathcal{R} for notations.)

A full revolution, or 360° rotation about point O , rotates any point P around to itself so that $P' = P$. The diagram at the left below shows a rotation of 390° about O . Since 390° is 30° more than one full revolution, the image of any point P under 390° rotation is the same as its image under 30° rotation, and the two rotations are said to be equal. Similarly, the diagram at the right below shows that a 90° counterclockwise rotation is equal to a 270° clockwise rotation because both have the same effect on any point P .



$$\mathcal{R}_{O, 390} = \mathcal{R}_{O, 30}; \text{ Note: } 390 - 360 = 30 \quad \mathcal{R}_{O, 90} = \mathcal{R}_{O, -270}; \text{ Note: } 90 - 360 = -270$$

Activity 1

Share with your classmates five (5) uses of geometry that you have observed in Senegal.

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